Type decomposition of ideals in reduced groupoid C^*-algebras

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1. **Introduction**
2. Main results
3. Applications and Examples

Definition

A C^*-**algebra** is a $*$-subalgebra of $\mathfrak{B}(\mathcal{H})$ which is closed under the norm topology. Here $\mathcal{H}$ is a Hilbert space and $\mathfrak{B}(\mathcal{H})$ is the set of all linear continuous operators on $\mathcal{H}$.

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- There is an abstract definition of C^*-algebra, *i.e.*, a C^*-algebra is a $*$-algebra $\mathfrak{A}$ over $\mathbb{C}$ together with a norm $\|\cdot\|$ satisfying: $\forall a, b \in \mathfrak{A}$, we have
  1. $\|ab\| \leq \|a\| \cdot \|b\|$;
  2. $\|a^*a\| = \|a\|^2$ (C^*-identity);
  3. $\mathfrak{A}$ is complete under the norm $\|\cdot\|$.

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▶ For a locally compact Hausdorff space X, then

$$C_0(X) = \{f \in C(X) : f \text{ vanishes at infinity}\}$$

is a C^*-algebra. Conversely, any commutative C^*-algebra has this form.

- Let G be a discrete group. $C_c(G)$ is a $*$-algebra: For $f, g \in C_c(G)$ and $\gamma \in \mathcal{G}$, define

$$(f * g)(\gamma) := \sum_{\alpha \in G} f(\gamma\alpha^{-1})g(\alpha),$$
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- Consider the left regular representation $\lambda : C_c(G) \to \mathfrak{B}(\ell^2(G))$ by

$$\big(\lambda(f)\xi\big)(\gamma) := \sum_{\alpha \in G} f(\gamma\alpha^{-1})\xi(\alpha), \quad \text{where } f \in C_c(G) \text{ and } \xi \in \ell^2(G).$$

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- The **reduced norm** on $C_c(G)$ is $\|f\|_r := \|\lambda(f)\|$. The **reduced group** C^*-**algebra** $C_r^*(G)$ is the completion of the $*$-algebra $C_c(G)$ w.r.t. $\|\cdot\|_r$.

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Example

For $G = \mathbb{Z}$, then $C_r^*(G) = C(\mathbb{T})$ by the Fourier transformation.

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- In 1980s, Renault initiated the study of C^*-algebras associated to groupoids. Since then, groupoids gave rise to an extraordinarily rich class of operator algebras.

- Recently in 2020, X.Lin proved that every classifiable C^*-algebra admits a (twisted) groupoid model.

Basic notions for groupoids

- A **groupoid** consists of a set $\mathcal{G}$, a subset $\mathcal{G}^{(0)}$ called the **unit space**, two maps $s, r : \mathcal{G} \to \mathcal{G}^{(0)}$ called the **source** and **range** maps, respectively, a **composition law**:

$$\mathcal{G}^{(2)} := \{(\gamma_1, \gamma_2) \in \mathcal{G} \times \mathcal{G} : s(\gamma_1) = r(\gamma_2)\} \ni (\gamma_1, \gamma_2) \mapsto \gamma_1 \gamma_2 \in \mathcal{G},$$

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- For $U \subseteq \mathcal{G}^{(0)}$, denote $\mathcal{G}_U := s^{-1}(U)$. Étaleness $\Rightarrow$ each $\mathcal{G}_x := \mathcal{G}_{\{x\}}$ is discrete.

Reduced groupoid C^*-algebras

- Let $\mathcal{G}$ be a locally compact, Hausdorff and étale groupoid.
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- For $x \in \mathcal{G}^{(0)}$, consider $\lambda_x : C_c(\mathcal{G}) \to \mathfrak{B}(\ell^2(\mathcal{G}_x))$ defined by

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- The **reduced norm** on $C_c(\mathcal{G})$ is $\|f\|_r := \sup_{x \in \mathcal{G}^{(0)}} \|\lambda_x(f)\|$. The **reduced groupoid C^*-algebra** $C_r^*(\mathcal{G})$ is the completion of the $*$-algebra $C_c(\mathcal{G})$ with respect to the reduced norm $\|\cdot\|_r$.

Example

A group G is a groupoid with unit space $\{1_G\}$. A set X is a groupoid with unit space X, $r(x) = s(x) = x$.

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Example (Pair groupoid)

For a set X, the product $X \times X$ is a groupoid with unit space X, $s(x,y) = y$, $r(x,y) = x$, $(x,y) \cdot (y,z) = (x,z)$ and $(x,y)^{-1} = (y,x)$.

- $C_r^*(X \times X)$ is the algebra of compact operators on $\ell^2(X)$ if X is discrete.

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Example (Transformation groupoid)

Let Γ be a group acting on a set X. The product $X \times \Gamma$ is a groupoid with unit space X, $s(x, \gamma) = \gamma^{-1}x$, $r(x, \gamma) = x$, $(x, \gamma)^{-1} = (\gamma^{-1}x, \gamma^{-1})$ and $(\gamma x, \gamma) \cdot (x, \gamma') = (\gamma x, \gamma\gamma')$. Denote this groupoid by $X \rtimes \Gamma$.

- $C_r^*(X \times \Gamma) \cong C(X) \rtimes_r \Gamma$ if X is compact Hausdorff and Γ is discrete.

Ideal Structures

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- An important tool to study $C_r^*(\mathcal{G})$ is to study its **ideal structure**, which is invariant under Morita equivalence.

- If we understand the ideal structure, then we can chop a C^*-algebra into easy-handled pieces. This philosophy plays an important role not only in C^*-algebra structure theory, but also in higher index theory.

Known results

- Simplicity of $C_r^*(G)$: Breuillard, Kalantar, Kennedy, Ozawa...
  (dynamics of G on its Furstenberg boundary)
- Ideal structure of $C(X) \rtimes_r \Gamma$: Renault, Sierakowski,...
  (invariant open subsets of X and dynamics)
- Ideal structure of $C_u^*(X)$: Chen, Wang, Z,...
  (coarse geometry of metric spaces)
- Ideal structure of $C_r^*(\mathcal{G})$: Li, Bönicke, Brix, Carlsen, Sims, Brown, Fuller, Pitts, Reznikoff, ...

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Convention

Always assume that $\mathcal{G}$ is a locally compact Hausdorff and étale groupoid.

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Inner supports

Idea: Using commutative objects to parametrise ideals in $C_r^*(\mathcal{G})$.

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  $\Longrightarrow I \cap C_0(\mathcal{G}^{(0)}) = C_0(U_I)$ for

$$U_I := \{x \in \mathcal{G}^{(0)} : \exists\ f \in I \cap C_0(\mathcal{G}^{(0)}) \text{ s.t. } f(x) \neq 0\}.$$

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$$U_I := \{x \in \mathcal{G}^{(0)} : \exists\ f \in I \cap C_0(\mathcal{G}^{(0)}) \text{ s.t. } f(x) \neq 0\}.$$

Then U_I is open and **invariant**: $\forall \gamma \in \mathcal{G}$ with $s(\gamma) \in U_I$, then $r(\gamma) \in U_I$.

- We say that U_I is the **inner support** of I.

Dynamical Ideals

Definition

Given an invariant open $U \subseteq \mathcal{G}^{(0)}$, the ideal in $C_r^*(\mathcal{G})$ generated by $C_0(U)$ is called the **dynamical ideal** associated to U, denoted by $I(U)$.

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$$\{\text{dynamical ideals}\} \longleftrightarrow \{\text{invariant open subsets of } \mathcal{G}^{(0)}\}.$$

Characterising all ideals being dynamical

Theorem (Bönicke and Li, 2020)

All ideals in $C_r^*(\mathcal{G})$ are dynamical $\Longleftrightarrow$ $\mathcal{G}$ is inner exact and has the residual intersection property.

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- $\mathcal{G}$ is **inner exact** if for any invariant open $U \subseteq \mathcal{G}^{(0)}$, the following sequence is exact:

$$0 \longrightarrow C_r^*(\mathcal{G}_U) \longrightarrow C_r^*(\mathcal{G}) \longrightarrow C_r^*(\mathcal{G}_{\mathcal{G}^{(0)} \setminus U}) \longrightarrow 0.$$

(Note: $C_r^*(\mathcal{G}_U) \cong I(U)$.)

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- $\mathcal{G}$ has the **intersection property** if $\forall$ ideal $I \neq 0$, then $I \cap C_0(\mathcal{G}^{(0)}) \neq 0$. $\mathcal{G}$ has the **residual intersection property** if $\mathcal{G}_{\mathcal{G}^{(0)} \setminus U}$ has the intersection property for any invariant open $U \subseteq \mathcal{G}^{(0)}$.

- Regard elements in $C_r^*(\mathcal{G})$ as functions on $\mathcal{G}$: $\exists$ a linear contractive map

$$j : C_r^*(\mathcal{G}) \to C_0(\mathcal{G}) \quad \text{by} \quad j(a)(\gamma) := \langle \lambda_{s(\gamma)}(a)\delta_{s(\gamma)}, \delta_\gamma \rangle, \quad \text{for} \quad \gamma \in \mathcal{G}.$$

(**Intuition:** $j|_{C_c(\mathcal{G})} = \mathrm{Id}_{C_c(\mathcal{G})}$.)

Outer supports and a sandwiching result

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- Given an ideal I in $C_r^*(\mathcal{G})$, define its **outer support** to be

$$\begin{aligned} V_I : &= \{x \in \mathcal{G}^{(0)} : \exists\ a \in I \text{ such that } j(a)(x) \neq 0\} \\ &= \bigcup_{a \in I} r(\{\gamma \in \mathcal{G} : j(a)(\gamma) \neq 0\}). \end{aligned}$$

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Theorem (Brix, Carlsen and Sims, 2024)

Assume that $\mathcal{G}$ is inner exact. Given an ideal I in $C_r^*(\mathcal{G})$, then $I(U_I)$ is the largest dynamical ideal contained in I, and $I(V_I)$ is the smallest dynamical ideal containing I.

A sandwiching result: beyond inner exactness

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Definition (Li and Z, 2025+)

For an invariant open subset $U \subseteq \mathcal{G}^{(0)}$, the associated **ghostly ideal** is:

$$\tilde{I}(U) := \{a \in C_r^*(\mathcal{G}) : r(\{\gamma \in \mathcal{G} : j(a)(\gamma) \neq 0\}) \subseteq U\}.$$

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Proposition

For any invariant open $U \subseteq \mathcal{G}^{(0)}$, we have a short exact sequence:

$$0 \longrightarrow \tilde{I}(U) \longrightarrow C_r^*(\mathcal{G}) \longrightarrow C_r^*(\mathcal{G}_{\mathcal{G}^{(0)} \setminus U}) \longrightarrow 0.$$

Type decomposition

- Two types of ideals:
  1. **Type I**: $I(U_I) \subseteq I \subseteq I(V_I)$ (no need for the ghostly part!)
  2. **Type II**: $U_I = V_I$, *i.e.*, $I(U_I) \subseteq I \subseteq \tilde{I}(U_I)$.

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Theorem (Li and Z, 2025+)

Any ideal in $C_r^*(\mathcal{G})$ can be reconstructed using type I and type II ideals.

Question

When is a given ideal I in $C_r^*(\mathcal{G})$ of type II, *i.e.*, $U_I = V_I$?

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- For $x \in \mathcal{G}^{(0)}$, the isotropy group at x is $\mathcal{G}_x^x := \{\gamma \in \mathcal{G} : s(\gamma) = r(\gamma) = x\}$, and the isotropy groupoid is

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- $\mathcal{G}$ is called **effective** if the interior of $\mathrm{Iso}(\mathcal{G})$ is $\mathcal{G}^{(0)}$. $\mathcal{G}$ is called **strongly effective** if $\mathcal{G}_D$ is effective for any invariant closed $D \subseteq \mathcal{G}^{(0)}$.

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Theorem (Li and Z, 2025+)

Let $\mathcal{G}$ be a locally compact Hausdorff and étale groupoid. An ideal I in $C_r^*(\mathcal{G})$ is of type II (i.e., $U_I = V_I$) if and only if $\mathcal{G}_{V_I \setminus U_I}$ is effective. Hence if $\mathcal{G}$ is strongly effective, then every ideal in $C_r^*(\mathcal{G})$ is of type II.

Theorem (Atiyah-Singer, 1963)

Let M be a closed smooth manifold, and E, F be smooth vector bundles over M. Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator. Then P is Fredholm, and its analytical index is equal to its topological index:

$$\mathrm{Index}_a(P) = \mathrm{Index}_{top}(P).$$

- $\mathrm{Index}_a(P) := \dim(KerP) - \dim(CokerP)$, called the **Fredholm index**.

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▶ Higher index theory: studying index theory on non-compact manifolds.

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Roe algebras

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- Let X be a discrete metric space of bounded geometry (*i.e.*, for any $R > 0$, $\sup_{x \in X} \sharp B(x,R) < \infty$).

- An operator $T \in \mathfrak{B}(\ell^2(X))$ can be regarded as an X-by-X matrix $(T_{x,y})_{x,y \in X}$ where $T_{x,y} \in \mathbb{C}$. Say that T has **finite propagation** if $\exists\ R > 0$ such that $T_{x,y} = 0$, $\forall\ x, y \in X$ with $d(x,y) > R$.

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Definition

Denote by $\mathbb{C}_u[X]$ the $*$-algebra in $\mathfrak{B}(\ell^2(X))$ consisting of all finite propagation operators. The **uniform Roe algebra** of X is the norm closure of $\mathbb{C}_u[X]$ in $\mathfrak{B}(\ell^2(X))$, denoted by $C_u^*(X)$.

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- (Uniform) Roe algebras play an important role in index theory, C^*-algebra theory, operator theory, topological dynamics, etc.

- Let (X, d) be a discrete metric space with bounded geometry. For $r > 0$, denote $E_r := \{(x, y) \in X \times X : d(x, y) \leq r\}$. As a topological space,

$$G(X) := \bigcup_{r>0} \overline{E_r}^{\beta(X \times X)} \subseteq \beta(X \times X),$$

where $\beta(X \times X)$ is the Stone-Čech compactification of $X \times X$.

Our motivation: the coarse groupoid

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- Consider $(r, s) : G(X) \to \beta X \times \beta X$, which is injective. Hence $G(X)$ can be endowed with a groupoid structure induced by the pair groupoid $\beta X \times \beta X$, called the **coarse groupoid** of X.

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Proposition (Skandalis-Tu-Yu, 2002)

The coarse groupoid $G(X)$ for a discrete metric space X with bounded geometry is étale, locally compact and Hausdorff with unit space βX.

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- $C_r^*(G(X)) \cong C_u^*(X)$, the uniform Roe algebra.

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Recall: $T \in C_u^*(X)$ is called a ghost operator if $\forall\ \varepsilon > 0$, $\exists$ finite $F \subseteq X$ such that $|T_{x,y}| < \varepsilon$ for $(x,y) \notin F \times F$, where $T_{x,y} := \langle \delta_x, T\delta_y \rangle \in \mathbb{C}$.

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Theorem (Chen-Wang 2004, Roe-Willett 2014)

For a discrete metric space X of bounded geometry, TFAE:

1. *X has Yu's Property A;*
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- Since $G(X)$ is principal, then all ideals in $C_u^*(X)$ have type II.

Regular ideals

- Let A be a C^*-algebra and $X \subseteq A$. Denote

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For a topological space X and open $U \subseteq X$, $C_0(U)$ is regular $\Longleftrightarrow$ the interior of $\overline{U}$ equals U (in this case, we say that U is regular).

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Lemma

For a regular open $U \subseteq \mathcal{G}^{(0)}$, we have $\tilde{I}(U) = I(U)^{\perp\perp}$.

Regular ideals

- $\mathcal{G}$ has the **regular intersection property** if for any regular ideal I in $C_r^*(\mathcal{G})$, $I \cap C_0(\mathcal{G}^{(0)}) = 0$ implies $I = 0$.

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Theorem (Li and Z, 2025+)

TFAE:

1. *every regular ideal is ghostly;*
2. $U \mapsto \tilde{I}(U)$ *is a bijection between regular invariant open subsets of* $\mathcal{G}^{(0)}$ *and regular ideals in* $C_r^*(\mathcal{G})$*;*
3. *for any regular ideal* I, $U_I = V_I$*;*
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- (4) $\Rightarrow$ (2) is proved by Brown, Fuller, Pitts and Reznikoff (2024).

Groups acting on its Furstenberg boundary

- Let G be a discrete group, and X a G-boundary (*i.e.*, a minimal and strongly proximal compact Hausdorff space).

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- For $x \in X$, denote $G_x := \{g \in G : gx = x\}$. If G_x is amenable, we have the trivial representation $\tau_x : C_r^*(G_x) \to \mathbb{C}$. Denote its kernel by I_x.

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- The **Furstenberg boundary** $\partial_F G$ is the universal G-boundary. $\forall x \in \partial_F G$, G_x is amenable. There is a c.c.p. map

$$\vartheta_{x,r} : C(\partial_F G) \rtimes_r G \to C_r^*(G_x),$$

and denote the induced ideal

$$\mathrm{Ind}(I_x) := \{x \in C(\partial_F G) \rtimes_r G : \vartheta_{x,r}(b^* a^* ab) \in I_x, \forall\ b \in C(\partial_F G) \rtimes_r G\}.$$

Theorem (Li and Z, 2025+)

Consider a discrete group G acting on its Furstenberg boundary $\partial_F G$. Given $x \in \partial_F G$, TFAE:

1. *the ideal $\mathrm{Ind}(I_x)$ in $C(\partial_F G) \rtimes_r G$ is ghostly;*
2. $V_{\mathrm{Ind}(I_x)} \neq \partial_F G$;
3. $\mathrm{Ind}(I_x) = 0$;
4. *the action is (topologically) free.*

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Corollary

TFAE:

1. *the Thompson's group F is non-amenable;*
2. *the Thompson's group T is C^*-simple;*
3. *the action of T on $\partial_F T$ is (topologically) free;*
4. $\exists x \in \partial_F T$ *such that* $\mathrm{Ind}(I_x)$ *is ghostly;*
5. $\exists x \in \partial_F T$ *such that* $V_{\mathrm{Ind}(I_x)} = \emptyset$.

Thank you for your attention!