The de Rham cohomology of a Lie group modulo a dense subgroup*

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Abstract: Let H be a dense subgroup of a Lie group G with Lie algebra $\mathfrak{g}$. We show that the (diffeological) de Rham cohomology of G/H equals the Lie algebra cohomology of $\mathfrak{g}/\mathfrak{h}$, where $\mathfrak{h}$ is the ideal $\{Z \in \mathfrak{g} : \exp(tZ) \in H \text{ for all } t \in \mathbf{R}\}$.

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1. Diffeological de Rham complex

Let us call *ordinary* the k-forms on Euclidean open sets and manifolds and operations on them (exterior derivative d, pull-back ϕ^*).

Definitions (Souriau 1985)

Let X and Y be diffeological spaces.

(a) A (diffeological) ***k-form*** β on Y is a functional which sends each plot $Q : V \to Y$ to an ordinary k-form on V, *denoted* $Q^\star\beta$ (note special $\star$). As compatibility, we require: if $\phi \in C^\infty(U, V)$ (so $Q \circ \phi$ is another plot), then

$$(Q \circ \phi)^\star\beta = \phi^* Q^\star\beta, \qquad \phi^* : \text{ordinary pull-back.}$$

(b) Its ***pull-back*** $F^*\beta$ by a smooth map $F : X \to Y$ is the k-form on X defined by: if P is a plot of X (so $F \circ P$ is a plot of Y), then

$$P^\star F^*\beta = (F \circ P)^\star\beta, \qquad F^* : \text{being defined.}$$

(c) Its ***exterior derivative*** $d\beta$ is the $(k + 1)$-form defined by: if Q is a plot of Y, then $Q^\star d\beta = d[Q^\star\beta]$, with ordinary d on the right-hand side.

1. Diffeological de Rham complex

- The *de Rham complex* $(\Omega^\bullet(Y), d)$ is the sum over k of the spaces $\Omega^k(Y)$ of k-forms on Y, endowed with the differential (c), which satisfies $d^2 = 0$ since the ordinary d does. Its cohomology is the *de Rham cohomology* $H^\bullet_{dR}(Y)$.
- (a,b,c) easily imply, for all k-forms β and smooth maps F, G:

$$(F \circ G)^*\beta = G^*F^*\beta, \qquad d[F^*\beta] = F^*d\beta. \qquad \spadesuit$$

- If Y is a manifold and $\beta \in \Omega^k(Y)$, applying (a) to *charts* $V \to Y$ implies that there is a unique ordinary k-form b such that $Q^\star\beta$ and $Q^\star d\beta$ are always the ordinary Q^*b and Q^*db.
- So we may (and will) confuse ordinary k-forms and operations with diffeological ones. Then we may retire the special $\star$, so that (a,b,c) become special cases of ♠.

2. Souriau's criterion

This criterion tells when a given k-form is pulled back from a quotient.

Theorem (Souriau's criterion, 1985)

Let $s : X \to Y$ be a subduction and $\alpha \in \Omega^k(X)$. In order that $\alpha = s^*\beta$ for some $\beta \in \Omega^k(Y)$, it is necessary and sufficient that all pairs of plots P, Q of X satisfy:

$$s \circ P = s \circ Q \quad \Rightarrow \quad P^*\alpha = Q^*\alpha.$$

Moreover β is then unique, i.e. pull-back $s^* : \Omega^k(Y) \to \Omega^k(X)$ is injective.

3. Differential forms on $X = G/H$

Our goal is:

Theorem

Let H *be a dense subgroup of a Lie group* G *with Lie algebra* $\mathfrak{g}$. *Then* $\mathfrak{h} := \{Z \in \mathfrak{g} : \exp(tZ) \in H$ *for all* $t \in \mathbf{R}\}$ *is an ideal in* $\mathfrak{g}$, *and giving* $X = G/H$ *the quotient diffeology, we have isomorphisms*

$$(\Omega^\bullet(X), d) = (\textstyle\bigwedge^\bullet(\mathfrak{g}/\mathfrak{h})^*, d) \quad \textit{and hence} \quad H^\bullet_{\mathrm{dR}}(X) = H^\bullet(\mathfrak{g}/\mathfrak{h}).$$

Here the right-hand sides are the Chevalley-Eilenberg complex of $\mathfrak{g}/\mathfrak{h}$ and its cohomology, whose definitions we will review during the proof.

Sketch of proof. H is canonically a Lie group, with Lie algebra as above: see Bourbaki, who in effect show (H, subset diffeology) *is a manifold* in our sense. Then, as H is dense and normalizers are closed, we have

$$G \text{ normalizes } \mathfrak{h}: \quad g.\mathfrak{h}.g^{-1} = \mathfrak{h} \quad \text{for all } g \in G. \qquad \heartsuit$$

Deriving this at e one obtains that $\mathfrak{h}$ is an ideal, i.e. $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

3. Differential forms on $X = G/H$

The core of the proof is the next proposition, where we will write:

- $\Pi : G \to X$ for the natural projection, $\Pi(q) = qH$,
- $L_g : G \to G$ for left translation, $L_g(q) = gq$,
- $R_g : G \to G$ for right translation, $R_g(q) = qg$,
- $g.v := DL_g(q)(v)$ and $v.g := DR_g(q)(v)$, whenever $v \in T_qG$.

Proposition

Pull-back via Π *defines a bijection* Π^* *from* $\Omega^k(X)$ *onto the set of those* $\mu \in \Omega^k(G)$ *that are*

(a) right-invariant: $R_g^*\mu = \mu$ *for all* $g \in G$;

(b) $\mathfrak{h}$-horizontal: $\mu(Z_1, \ldots, Z_k) = 0$ *whenever one of the* $Z_j \in \mathfrak{g}$ *is in* $\mathfrak{h}$.

Proof. Suppose $\mu = \Pi^*\alpha$ for some $\alpha \in \Omega^k(X)$. Let us prove (a):

- $\Pi \circ R_h = \Pi$ implies $R_h^*\Pi^*\alpha = \Pi^*\alpha$ for all $h \in H$.
- As H is dense, the same holds for all $g \in G$; so μ is right-invariant.

Now let us prove (b):

- Consider the plots $P, Q : \mathfrak{g} \times \mathfrak{h} \to G$ sending $u = (Z, W)$ to

$$P(u) = \exp(Z), \quad \text{resp.} \quad Q(u) = \exp(Z)\exp(W).$$

  (To get *literal* plots, use bases to identify $U := \mathfrak{g} \times \mathfrak{h}$ with $\mathbf{R}^m$.)
- Clearly $\Pi \circ P = \Pi \circ Q$. So by Souriau's criterion $P^*\mu = Q^*\mu$.
- Which when evaluated on vectors $(Z_i, W_i) \in T_{(0,0)}U$ yields

$$\mu(Z_1, \ldots, Z_k) = \mu(Z_1 + W_1, \ldots, Z_k + W_k),$$

  whence (b) by choosing $W_j = -Z_j$.

Conversely, let $\mu \in \Omega^k(G)$ satisfy (a) and (b), and let $P, Q : U \to G$ be any two plots with $\Pi \circ P = \Pi \circ Q$. We must show that $P^*\mu = Q^*\mu$:

- Note $R(u) := P(u)^{-1}Q(u)$ defines a plot $R : U \to H$.

- Then $P \times Q \times R$ is an ordinary smooth map sending $u \in U$ to

$$(P(u), Q(u), R(u)) =: (g, gh, h).$$

- Its derivative at u sends each $\delta u \in T_u U$ to a tangent vector

$$D(P \times Q \times R)(u)(\delta u) =: (\delta g, \delta[gh], \delta h)$$

in $T_{(g,gh,h)}(G \times G \times H)$.

- Now $\delta[gh] = \delta g.\, h + g.\, \delta h$. Hence, given $\delta_1 u, \ldots, \delta_k u \in T_u U$,

$$\begin{aligned} \delta_i[gh].\,(gh)^{-1} &= [\delta_i g.\, h + g.\, \delta_i h].\,(gh)^{-1} \\ &= \delta_i g.\, g^{-1} + g.\, \delta_i h.\, h^{-1}.\, g^{-1}. \end{aligned}$$

◇

- As G normalizes $\mathfrak{h}$, the term $W_i := g.\, \delta_i h.\, h^{-1}.\, g^{-1}$ above is in $\mathfrak{h}$.

Thus

$$\begin{aligned}
(Q^*\mu)(\delta_1 u, \ldots, \delta_k u) &= \mu(\delta_1[gh], \ldots, \delta_k[gh]) \\
&= \mu(\delta_1[gh].(gh)^{-1}, \ldots, \delta_k[gh].(gh)^{-1}) && \text{by (a)} \\
&= \mu(\delta_1 g.g^{-1} + W_1, \ldots, \delta_k g.g^{-1} + W_k) && \text{by } \diamondsuit \\
&= \mu(\delta_1 g.g^{-1}, \ldots, \delta_k g.g^{-1}) && \text{by (b)} \\
&= \mu(\delta_1 g, \ldots, \delta_k g) && \text{by (a)} \\
&= (P^*\mu)(\delta_1 u, \ldots, \delta_k u).
\end{aligned}$$

Hence by Souriau's criterion μ is in the image of the injection Π^*. □

4. Passage to left-invariant forms

- Lie algebra cohomology is traditionally defined using *left*-, not right-invariant forms. So we need to pass from one to the other.
- For that we simply pull back by the inversion map, $\mathrm{inv}(g) = g^{-1}$. Indeed the relation $\mathrm{inv} \circ \mathrm{L}_g = \mathrm{R}_{g^{-1}} \circ \mathrm{inv}$ implies that $\mu \in \Omega^k(\mathrm{G})$ is right-invariant iff $\omega = \mathrm{inv}^*\mu$ is left-invariant. Also inv^* preserves $\mathfrak{h}$-horizontality, because $g \mapsto g^{-1}$ has derivative $\mathrm{Z} \mapsto -\mathrm{Z}$ at e. Thus we have:

Corollary

Pull-back via $\check{\Pi} := \Pi \circ \mathrm{inv}$ *defines a bijection* $\check{\Pi}^* = \mathrm{inv}^*\Pi^*$ *from* $\Omega^k(\mathrm{X})$ *onto the set of those* $\omega \in \Omega^k(\mathrm{G})$ *that are*

(a) left-invariant: $\mathrm{L}_g^*\omega = \omega$ *for all* $g \in \mathrm{G}$;

(b) $\mathfrak{h}$-horizontal: $\omega(\mathrm{Z}_1, \ldots, \mathrm{Z}_k) = 0$ *whenever one of the* $\mathrm{Z}_j \in \mathfrak{g}$ *is in* $\mathfrak{h}$.

Left-invariant forms make a subcomplex $(\Omega^\bullet(G)^G, d)$ of $(\Omega^\bullet(G), d)$ which depends only on $\mathfrak{g}$. Indeed $\omega \in \Omega^k(G)^G$ satisfies, for all $Z_i \in \mathfrak{g}$,

i) the relation $\omega(g.Z_1, \ldots, g.Z_k) = \omega(Z_1, \ldots, Z_k)$, which characterizes ω by its value at the identity, $\omega_e \in \bigwedge^k \mathfrak{g}^*$.

ii) the *Chevalley-Eilenberg formula*

$$d\omega(Z_0, \ldots, Z_k) = \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([Z_i, Z_j], Z_0, \ldots, \widehat{Z}_i, \ldots, \widehat{Z}_j, \ldots, Z_k)$$

which computes $(d\omega)_e$ from ω_e alone.

- Thus, taking this formula as the definition of a coboundary d on $\bigwedge^\bullet \mathfrak{g}^*$, we obtain a complex $(\bigwedge^\bullet \mathfrak{g}^*, d)$ isomorphic to $(\Omega^\bullet(G)^G, d)$ via $\omega \mapsto \omega_e$. Its cohomology is by definition the *Lie algebra cohomology* $H^\bullet(\mathfrak{g})$.
- We study the subcomplex $\Omega^\bullet(G)^G_\mathfrak{h}$ of forms that are also $\mathfrak{h}$-horizontal; or equivalently its image $(\bigwedge^\bullet \mathfrak{g}^*)_\mathfrak{h}$ defined inside $\bigwedge^\bullet \mathfrak{g}^*$ by the same property (b).

5. Passage to $\bigwedge^\bullet(\mathfrak{g}/\mathfrak{h})^*$

Recall that $\mathfrak{h}$ is an ideal, and let $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ be the natural projection.

Lemma (elementary)

Pull-back via π defines an isomorphism π^* from $(\bigwedge^\bullet(\mathfrak{g}/\mathfrak{h})^*, d)$ onto the subcomplex $((\bigwedge^\bullet\mathfrak{g}^*)_\mathfrak{h}, d)$ of $(\bigwedge^\bullet\mathfrak{g}^*, d)$.

Now, composing the three isomorphisms of complexes we have seen completes the proof of the theorem:

$$\begin{array}{ccc}
\Omega^\bullet(G)^G_\mathfrak{h} & \xrightarrow{\quad \omega \mapsto \omega_e \quad} & (\bigwedge^\bullet\mathfrak{g}^*)_\mathfrak{h} \\
\uparrow{\scriptstyle \check{\Pi}^*} & & \uparrow{\scriptstyle \pi^*} \\
\Omega^\bullet(X) & \dashrightarrow & \bigwedge^\bullet(\mathfrak{g}/\mathfrak{h})^*.
\end{array}$$

□

6. Examples

We shall detail our theorem's content in the extreme cases where H is dense and either ***D-discrete*** or ***D-connected***, i.e. discrete or connected in its Lie group topology (= D-topology of its subset diffeology).

Corollary 1

Suppose the dense subgroup $H \subset G$ *is D-discrete* (*a.k.a. totally arcwise disconnected*: *arc components are singletons*). *Then we have*

$$H^\bullet_{dR}(G/H) = H^\bullet(\mathfrak{g}).$$

Moreover every Lie algebra cohomology ring $H^\bullet(\mathfrak{g})$ *occurs in this way.*

Proof.

- The a.k.a. is because the D-topology's connected components are the subset topology's *arc components* (Yamabe's theorem, see e.g. Hilgert–Neeb 2012).
- The formula is our theorem, since $\mathfrak{h} = \{0\}$.

- The "Moreover" is because a connected Lie group G always has countable dense subgroups H (Gelander–Le Maître 2017), and countable implies D-discrete (Iglesias-Zemmour 2013). □

Remarks:

(a) Thus e.g. $H^\bullet(\mathfrak{so}_3) = H^\bullet_{dR}(SO_3(\mathbf{R})/SO_3(\mathbf{Q}))$.

(b) *Uncountable* dense D-discrete subgroups also exist, e.g. in any connected nilpotent Lie group G (de Saxcé 2013). Our formula still covers those.

(c) When $G = V$ is the additive group of a vector space and $H = \Lambda$ a dense D-discrete additive subgroup, the Chevalley–Eilenberg coboundary vanishes and we obtain a full exterior algebra,

$$H^\bullet_{dR}(V/\Lambda) = \bigwedge^\bullet V^*,$$

as was already proved by Iglesias-Zemmour (2013).

Corollary 2

Suppose the dense subgroup $H \subset G$ is D-connected (a.k.a. arcwise connected). Then $\mathfrak{g}/\mathfrak{h}$ is abelian and we have

$$H^\bullet_{dR}(G/H) = \bigwedge^\bullet(\mathfrak{g}/\mathfrak{h})^*.$$

Moreover the resemblance to the last remark is no accident: indeed, we can always rewrite G/H as a ***quasitorus*** *V/Λ, where $V = \mathfrak{g}/\mathfrak{h}$ and Λ is a countable dense additive subgroup.*

Proof.

- The a.k.a. is again by Yamabe's theorem.
- Commutativity of $\mathfrak{g}/\mathfrak{h}$ is a theorem of van Est (1951), also found in Bourbaki.
- The formula is our theorem, since the Chevalley–Eilenberg coboundary vanishes.
- To see "Moreover" we build the following commutative diagram:

6. Examples

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta & \longrightarrow & \Gamma & \longrightarrow & \Lambda & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \tilde{H} & \longrightarrow & \tilde{G} & \longrightarrow & V & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & X & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

- Row 3 defines X as the diffeological quotient G/H; recall that H is normal by ♡, and G is connected as the closure of H.
- On row 2, let $\tilde{G} :=$ the universal covering of G, $\tilde{H} :=$ its integral subgroup with Lie algebra $\mathfrak{h}$, and $V := \tilde{G}/\tilde{H}$.
- Then $\tilde{H}$ is closed, and $\tilde{H}$ and V are simply connected (Bourbaki). In particular $V = \mathfrak{g}/\mathfrak{h}$, as the unique simply connected Lie group with that abelian Lie algebra.

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \vdots & & \\
1 & \longrightarrow & \Delta & \longrightarrow & \Gamma & \longrightarrow & \Lambda & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \vdots & & \\
1 & \longrightarrow & \tilde{H} & \longrightarrow & \tilde{G} & \longrightarrow & V & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \vdots & & \\
1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & X & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \vdots & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

- On row 1, let $\Gamma := \ker(\tilde{G} \to G)$, $\Delta := \Gamma \cap \tilde{H}$, and $\Lambda := \Gamma/\Delta$. These are countable (Hilgert–Neeb), and discrete in every sense.
- The exact sequences $\to$ are by construction *D-exact*: the subgroup and quotient in each have the subset and quotient diffeology.
- Then the nine lemma of Souriau (1985) says the diagram has a unique commutative completion by a sixth *D-exact* sequence $\dashrightarrow$: in other words X is also the diffeological quotient V/Λ. □

Final remark:

Our theorem does admit further examples where H is dense but neither D-discrete nor D-connected. As a simple one, consider for irrational α the subgroup

$$\mathrm{H} = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & \pm e^{i\alpha t} \end{pmatrix} : t \in \mathbf{R} \right\} = \mathrm{H}^+ \sqcup \mathrm{H}^-$$

of the 2-torus $\mathbf{T}^2$, which has two D-components $\mathrm{H}^\pm$, yet is connected because its already dense subgroup H^+ is. (In this case Corollary 2's conclusion $H^\bullet_{dR}(\mathrm{G}/\mathrm{H}) = \bigwedge^\bullet(\mathfrak{g}/\mathfrak{h})^*$ still holds with the same proof.)

End!