Example of Singular Reduction in Symplectic Diffeology

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The Context

Symplectic geometry There are two situations in which we need to improve symplectic geometry:

- Symplectic reduction in the presence of singularities
- Infinite dimensional “symplectic” spaces

The reduction is defined on presymplectic manifolds or co-isotropic submanifolds. This is well documented when the reduction is regular, that is, when it does not involve singularities such that the reduced space is itself a manifold.

For infinite “symplectic” spaces, what we have is a series of examples and heuristic constructions.

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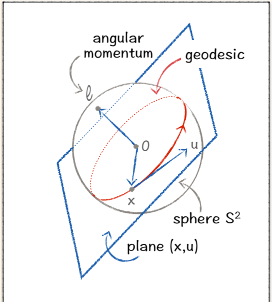

The geodesics of the sphere

$$\begin{cases} (x,u) \in US^2, \\ x, u \in S^2 \text{ and } \langle x, u\rangle = 0. \\ \ell = x \wedge u. \end{cases}$$

- The space of geodesics: $\mathrm{Geod}(S^2) = \{\ell\} = S^2$, is a manifold.
- The symplectic struture: $\omega(\delta\ell, \delta'\ell) \propto \langle \ell, \delta\ell \wedge \delta'\ell\rangle$.

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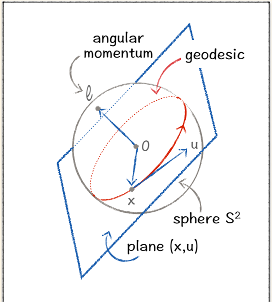

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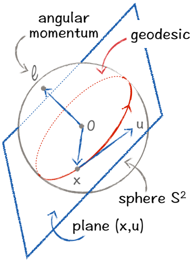

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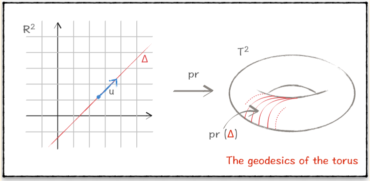

- The space of geodesics: $\mathrm{Geod}(T^2)$ is fibered over S^1 with fiber over u: the torus (rational or irrational) of slope u.

$$\mathrm{Geod}(T^2) = \{(u, \tau) \mid u = (\cos(\theta), \sin(\theta)), \theta \in \mathbf{R}, \tau \in \mathbf{R}/\cos(\theta)\mathbf{Z} + \sin(\theta)\mathbf{Z}\}.$$

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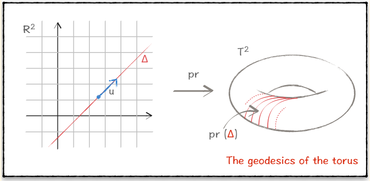

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Claim: The space Geod(T^2) is not a manifold because of irrational tori, when $\cos(\theta)$ and $\sin(\theta)$ are independent over $\mathbf{Q}$. But, as a diffeological space:

Proposition:[1] The space Geod(T^2), quotient of the unit tangent bundle UT^2, is 2-dimensional and admits a parasymplectic form (a closed 2-form), projection of the canonical presymplectic form on UT^2.

Note: As a differential space (Sikorski, Frölicher…), quotient of the unit tangent bundle UT^2, Geod(T^2) is 1-dimensional equivalent to the circle S^1, because of the irrational tori; and obviously has no non-zero 2-form.

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Summary

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1. Diffeology framework pushes the limits of symplectic structure beyond the classical boundaries. Spaces of geodesics that are usually symplectic when they are manifolds continue to host a natural parasymplectic structure even when they are no more manifods.
2. Spaces of geodesics are the most common examples of "symplectic reductions", generally with singularities. Diffeology shows how to handle such singular reductions in finite-dimensional context.
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The Next Step

Among irregular situations for symplectic reduction, two cases appear frequently in mathematics or in physics:

- The symplectic space is infinite dimensional, for example a sphere S^∞ in an infinite dimensional Hilbert space.
- The reduction has singularities, for example some orbits are infinite lines and other are circles.

We shall show how these questions continue to be solved in the framework of diffeology. It is a particular example of the construction of an infinite dimensional quasiprojective space, that mixes the two situations.

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What is a Diffeology, for those who don't know yet

A diffeology is a smooth structure defined by means of parametrizations:

- A parametrization in a set X is any map $P\colon U \to X$, defined on some open subset of some Euclidean space $\mathbf{R}^n$.

A diffeology on a set X is defined as a set $\mathcal{D}$ of parametrizations, called plots, that satisfies three axioms:

- Covering The set $\mathcal{D}$ contains the constant parametrizations.
- Locality A parametrization which belongs locally to $\mathcal{D}$ belongs globally to $\mathcal{D}$.
- Smooth Compatibility The composite of any element of $\mathcal{D}$ by a smooth parametrization of its domain belongs to $\mathcal{D}$.

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Category {Diffeology}

Diffeological spaces are the objects of the category {Diffeology}, whose morphisms are the smooth maps:

- A smooth map from a diffeological space X to another X', is any map $f\colon X \to X'$ such that $f \circ P \in \mathcal{D}'$ for all $P \in \mathcal{D}$.

Smooth maps are denoted by $\mathcal{C}^\infty(X, X')$.

The isomorphisms are called diffeomorphisms, they are bijective smooth maps as well as their inverse.

Category {Diffeology} is stable by any set theoretic operation:

- Sum $X = \coprod_i X_i$.
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Quotient Spaces

A striking and important construction is the quotient diffeology, for any kind of partition:

Let $\sim$ be any equivalence relation on a diffeological space X, that is, a partition of X. We can push forward the diffeology of X onto the quotient set $\mathrm{Q} = \mathrm{X}/\sim$, by the natural projection $\mathrm{class}\colon \mathrm{X} \to \mathrm{Q}$.

A plot of this quotient diffeology is any parametrization $\mathrm{P}\colon \mathrm{U} \to \mathrm{Q}$ such that everywhere:

$$\mathrm{P} \underset{\mathrm{loc}}{=} \mathrm{class} \circ \mathrm{R},$$

where R is some plot of X and the suffix loc means that R is required only locally.

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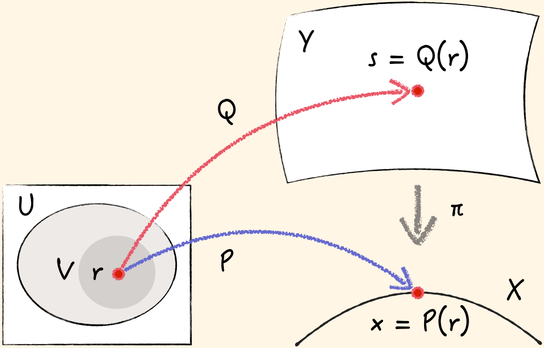
Subduction



$$P \mid V = \pi \circ Q$$

Differential Forms

A differential k-form α on a diffeological space X is defined by its pullbacks $P^*(\alpha)$ by the plots P in X.

Precisely, α is any mapping

$$\alpha\colon P \mapsto \alpha(P) \in \Omega^k(U),$$

for all plots

$$P : U \to X.$$

Satisfying, for all smooth parametrization $F\colon V \to U$, the following chain rule:

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$$\alpha(P \circ F) = F^*(\alpha(P)).$$

Differential Forms

A differential k-form α on a diffeological space X is defined by its pullbacks $P^*(\alpha)$ by the plots P in X.

Precisely, α is any mapping

$$\alpha \colon P \mapsto \alpha(P) \in \Omega^k(U),$$

for all plots

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Exterior Derivative

The exterior derivative of a k-form is given by:

$$\mathrm{d}\alpha(\mathrm{P}) = \mathrm{d}[\alpha(\mathrm{P})],$$

With

$$\mathrm{d}\colon \Omega^k(\mathrm{X}) \to \Omega^{k+1}(\mathrm{X}) \quad \text{and} \quad \mathrm{d} \circ \mathrm{d} = 0.$$

That defines a De Rham Complex for every diffeological space.

$$\mathrm{H}^k_{\mathrm{DR}}(\mathrm{X}) = \ker[\mathrm{d}\colon \Omega^k(\mathrm{X}) \to \Omega^{k+1}(\mathrm{X})]/\mathrm{d}[\Omega^{k-1}(\mathrm{X})].$$

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Let X and X′ be two diffeological spaces. Let

$$f\colon X \to X'$$

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Pushing Forms onto Quotients

Deciding if a differential form comes from a quotient is a recurrent question in diffeology.

Let X be a diffeological space and

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be a projection onto a quotient, and let $\alpha \in \Omega^k(X)$.

CRITERION There exists $\beta \in \Omega^k(Q)$ such that $\alpha = \pi^*(\beta)$ if and only if, for all plots P and P′ in X,

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Functional Diffeology on Complex Periodic Functions

Let X and X′ be two diffeological spaces, $\mathcal{C}^\infty(X, X')$ carries a natural diffeology called the functional diffeology.

The plots are the parametrizations $r \mapsto f_r$, defined on some Euclidean domain U such that

$$[(r, x) \mapsto f_r(x)] \in C^\infty(U \times X, X').$$

That diffeology makes the category Cartesian closed.

The space we will consider in the following is the space od complex periodic functions

$$C^\infty_{\text{per}}(\mathbf{R}, \mathbf{C}) = \{f \in C^\infty(\mathbf{R}, \mathbf{C}) \mid f(x + 1) = f(x)\},$$

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First, Fourier Transform

For all f in $C^\infty_{\text{per}}(\mathbf{R}, \mathbf{C})$, we associate the sequence of its Fourier coefficients $(f_n)_{n\in\mathbf{Z}}$

$$f_n = \int_0^1 f(x)e^{-2i\pi nx}\,dx,\ \forall n \in \mathbf{Z}.$$

The image of $f \mapsto (f_n)_{n\in\mathbf{Z}}$ is the vector space $\mathcal{E}$ of rapidly decreasing infinite complex series

$$\mathcal{E} = \Big\{(f_n)_{n\in\mathbf{Z}} \;\Big|\; f_n \in \mathbf{C}\ \&\ \forall p \in \mathbf{N},\ n^p f_n \xrightarrow[|n|\to\infty]{} 0\Big\}.$$

We push the functional diffeology on $C^\infty_{\text{per}}(\mathbf{R}, \mathbf{C})$ to $\mathcal{E}$. A plot $r \mapsto f_r$ will give a plot of $\mathcal{E}$

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Functional Diffeology on Fourier Coefficients – I

How to recognize a family $(f_n(r))_{n\in\mathbf{Z}}$ of smooth parametrizations in $\mathbf{C}$ coming from $C^\infty_{\mathrm{per}}(\mathbf{R},\mathbf{C})$?

THEOREM. A parametrizations $P : r \mapsto (f_n(r))_{n\in\mathbf{Z}}$ in $\mathcal{E}$ is a plot, for the pushforward of the functional diffeology on $C^\infty_{\mathrm{per}}(\mathbf{R},\mathbf{C})$, if and only if:

1. The functions $f_n : \mathrm{dom}(P) \to \mathbf{C}$ are smooth.
2. For all closed ball $\overline{\mathcal{B}} \subset \mathrm{dom}(P)$, for every $k \in \mathbf{N}$, for all $p \in \mathbf{N}$, there exists a positive number $M_{k,p}$ such that, for all integer $n \neq 0$,

$$\left|\frac{\partial^k f_n(r)}{\partial r^k}\right| \leq \frac{M_{k,p}}{|n|^p} \quad \text{for all } r \in \mathcal{B}. \qquad (\diamondsuit)$$

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Functional Diffeology on Fourier Coefficients – II

Remark 1. In other words, the parametrization $r \mapsto (f_n(r))_{n \in \mathbf{Z}}$ is a plot of this diffeology if the functions f_n are smooth and their derivatives are uniformly rapidly decreasing:

$$n^p \frac{\partial^k f_n(r)}{\partial r^k} \xrightarrow[|n| \to \infty]{} 0, \quad \text{for all } p \in \mathbf{N}.$$

Remark 2. By compactness, it is enough that, for every point $r_0 \in \mathrm{dom}(P)$, there exists a ball $\mathcal{B}'$ centered at r_0 such that $(\diamondsuit)$ holds to ensure that $(\diamondsuit)$ holds on every closed ball $\overline{\mathcal{B}} \subset \mathrm{dom}(P)$.

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The Infinite Torus

DEFINITION. Let T^∞ be the group of infinite sequences of complex unit number:

$$T^\infty = \prod_{n \in Z} U(1),$$

acting **C**-linearly on $\mathcal{E}$ by

$$(z_n)_{n\in Z} \cdot (Z_n)_{n\in Z} = (z_n Z_n)_{n\in Z}.$$

- A rapidly decreasing complex sequence is obviously transformed into another.
- Every element $z = (z_n)_{n\in Z} \in T^\infty$ is invertible,

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Action of The Infinite Torus

For every plot $r \mapsto (Z_n(r))_{n\in\mathbf{Z}}$ in $\mathcal{E}$, for all $p \in \mathbf{N}$,

$$\left|\frac{\partial^k z_n Z_n(r)}{\partial r^k}\right| = \left|z_n \frac{\partial^k Z_n(r)}{\partial r^k}\right| = \left|\frac{\partial^k Z_n(r)}{\partial r^k}\right|.$$

PROPOSITION. The action of $(z_n)_{n\in\mathbf{Z}}$ on $\mathcal{E}$ is smooth as well as its inverse, $(z_n)_{n\in\mathbf{Z}}$ acts on $\mathcal{E}$ by diffeomorphism. We got a monomorphism

$$\eta : T^\infty \to GL^\infty(\mathcal{E}) = GL(\mathcal{E}) \cap \mathrm{Diff}(\mathcal{E}).$$

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Diffeology of The Infinite Torus

DEFINITION. A tempered parametrization in T^∞ is a parametrization

$$\zeta : r \mapsto (z_n(r))_{n \in \mathbf{Z}}$$

that satisfies:

- The z_n are smooth and if for every $k \in \mathbf{N}$.
- For every r_0 in $\mathrm{dom}(\zeta)$, there exist a closed ball $\overline{\mathcal{B}} \subset \mathrm{dom}(\zeta)$ centered at r_0, a polynomial P_k and an integer N such that:

$$\forall r \in \mathcal{B}, \forall n > N, \quad \left| \frac{\partial^k z_n(r)}{\partial r^k} \right| \leq P_k(n).$$

PROPOSITION. The tempered parametrizations form a group diffeology on T^∞.

Diffeology of The Infinite Torus

Definition. A tempered parametrization in T^∞ is a parametrization

$$\zeta : r \mapsto (z_n(r))_{n\in\mathbf{Z}}$$

that satisfies:

- The z_n are smooth and if for every $k \in \mathbf{N}$.
- For every r_0 in $\mathrm{dom}(\zeta)$, there exist a closed ball $\overline{\mathcal{B}} \subset \mathrm{dom}(\zeta)$ centered at r_0, a polynomial P_k and an integer N such that:

$$\forall r \in \mathcal{B}, \forall n > N, \quad \left|\frac{\partial^k z_n(r)}{\partial r^k}\right| \leq P_k(n).$$

Proposition. The tempered parametrizations form a group diffeology on T^∞.

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Smooth Action of The Infinite Torus

PROPOSITION. Equipped with the tempered diffeology, the action of the group T^∞ on $\mathcal{E}$ is smooth. That is, the monomorphism $\eta : \mathrm{T}^\infty \to \mathrm{GL}^\infty(\mathcal{E})$ is smooth.

Next, for all $N \in \mathbf{N}$, let $\iota_N : \mathrm{T}^N \to \mathrm{T}^\infty$ be defined as follows:

$$\iota_N(z_n)_{n=1}^N = Z \quad \text{with} \quad \begin{cases} Z_n = z_n & \text{if } n \in \{1, \ldots, N\}, \\ Z_n = 1 & \text{otherwise.} \end{cases}$$

PROPOSITION. The smooth injection ι_N is a diffeomorphism from T^N onto its image equipped with the subset diffeology. That is, an induction.

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Induced Solenoids

Consider a sequence $\alpha = (\alpha_n)_{n\in\mathbb{Z}}$ of positive numbers, independent over $\mathbf{Q}$. That is,

$$\sum_{n\in\mathbb{Z}} q_n \alpha_n = 0 \quad \Rightarrow \quad q_n = 0,$$

for all finitely supported sequence of rational numbers $q_n \in \mathbf{Q}$.

In the following we will consider such sequences with $|\alpha_n| \leq 1$. Then, the map

$$\iota : \mathbf{R} \mapsto \mathrm{T}^\infty, \quad \text{defined by} \quad \iota(t) = \left(e^{2i\pi\alpha_n t}\right)_{n\in\mathbb{Z}},$$

which is obviously injective, is an induction, that is, a diffeomorphism onto its image equipped with the subset diffeology. We call the image $\iota(\mathbf{R}) \subset \mathrm{T}^\infty$, an irrational solenoid.

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Symplectic Structure on $C^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$

Let Surf be the standard symplectic form on $\mathbf{C}$:

$$\mathrm{Surf}_z(\delta z, \delta' z) = \frac{1}{2i}\left[\delta\bar{z}\,\delta' z - \delta'\bar{z}\,\delta z\right] \quad \forall z, \delta z, \delta' z \in \mathbf{C}.$$

The evaluation map: for all $x \in \mathbf{R}$, let

$$\hat{x} : C^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C}) \to \mathbf{C} \quad \text{with} \quad \hat{x}(f) = f(x), \quad \forall x \in \mathbf{R}.$$

Because $\hat{x}$ is smooth, the mean value of the pullback $\hat{x}^*(\mathrm{Surf})$ is a 2-form on $C^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$, and closed.

$$\omega = \frac{1}{\pi}\int_0^1 \hat{x}^*(\mathrm{Surf})\,dx \quad \text{with} \quad \begin{cases} \omega \in \Omega^2(C^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})) \\ d\omega = 0. \end{cases}$$

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Explicit value of ω: Let $P : r \mapsto f_r$ be a plot in $C^\infty_{\rm per}(\mathbf{R}, \mathbf{C})$,

$$\begin{aligned} \omega(P)_r(\delta r, \delta' r) &= \frac{1}{2i\pi} \int_0^1 \left\{ \frac{\partial \overline{f_r(x)}}{\partial r}(\delta r) \frac{\partial f_r(x)}{\partial r}(\delta' r) \right. \\ &- \left. \frac{\partial \overline{f_r(x)}}{\partial r}(\delta' r) \frac{\partial f_r(x)}{\partial r}(\delta r) \right\} dx. \end{aligned}$$

The Form ω is Indeed Symplectic

The closed 2-form ω is invariant by translation $\mathrm{Tr}_g : f \mapsto f + g$, and $C^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$ is an homogeneous space of itself. The moment map of this action is given, up to a constant, by:

$$\mu(f) = \frac{1}{2i\pi} d\left[g \mapsto \int_0^1 \bar{f}g - \bar{g}f\right].$$

PROPOSITION. The space $C^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$ as additive group acts transitively on itself by translation, preserving ω, and the moment map μ is injective. Thus $(C^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C}), \omega)$ is a diffeological symplectic space.

NOTE This is the definition given in the Introduction of "*Example of singular reduction in symplectic diffeology*" (P.I-Z). Proc. Amer. Math. Soc., 144(2):1309–1324 (2016).

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A Word on Moment Map

When we have a closed 2-form ω on a diffeological space X, invariant by a group G, there is a moment map

$$\mu\colon X \to \mathcal{G}^*/\Gamma,$$

where $\mathcal{G}^*$ is the space of momenta, that is, the spaces of left-invariant 1-forms on G, and Γ some representation of $\pi_1(X)$.

In the simplest case where $\omega = d\alpha$ and α is itself invariant:

$$\mu(x) = \hat{x}^*(\alpha) \in \mathcal{G}^*.$$

The case non invariant-exact is treated involving some diffeological constructions on the space Paths(X).

- "*The Moment Map in Diffeology*", P.I-Z. Memoir of the American Mathematical Society (2010) no.970, USA.

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The Moment Map of T^∞ on $\mathcal{E}$

The Moment Map μ of the (Hamiltonian and exact) action of T^∞ on $\mathcal{E}$:

$$\mu(Z) = \frac{1}{2i\pi} \sum_{n \in Z} |Z_n|^2 \, \pi_n^*(\theta) + \sigma,$$

- $\pi_n : T^\infty \to U(1)$ is the n-th projection $\pi_n(Z) = Z_n$,
- θ is the canonical invariant 1-form on $U(1)$,
- σ is some constant momentum of T^∞ (an invariant 1-form).

Note. For all plot $\zeta : r \mapsto (z_n(r))_{n \in Z}$ in T^∞, the value of the moment map $\mu(Z)(\zeta)_r(\delta r)$, which is defined as an infinite series, converges thanks to the definition of the tempered diffeology.

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The Moment Map of the Solenoid

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be the induced action of $\mathbf{R}$ on $\mathcal{E}$.

We call α-solenoid the subgroup

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Its moment map is given by reduction of μ:

$$\nu(Z) = h(Z)\,dt \quad \text{with} \quad h(Z) = \sum_{n\in\mathbf{Z}} \alpha_n |Z_n|^2 + c,$$

where c is some constant. The function h is called Hamiltonian.

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The Infinite Sphere and the Solenoid

Let S^∞_α be the unit level of the Hamiltonian h, for $c = 0$.

$$S^\infty_\alpha = \left\{ Z = (Z_n)_{n\in\mathbf{Z}} \in \mathcal{E} \;\middle|\; \sum_{n\in\mathbf{Z}} \alpha_n |Z_n|^2 = 1 \right\}.$$

Let QP^∞_α be the quotient of the inifinite ellipsoid S^∞_α by the action of the solenoid, equipped with the quotient diffeology, and pr be the projection,

$$\mathrm{pr} : S^\infty_\alpha \to QP^\infty_\alpha \quad \text{and} \quad QP^\infty_\alpha = S^\infty_\alpha / \underline{\mathbf{R}}.$$

We call the quotient space an: Infinite Quasiprojective Space, since it is a generalization of Prato's quasisphere [EP01].

The Infinite Sphere and the Solenoid

Let S^∞_α be the unit level of the Hamiltonian h, for $c = 0$.

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The Orbits of the Solenoid

The orbit of $Z = (Z_n)_{n \in \mathbf{N}} \in S^\infty_\alpha$ by the solenoid $\mathcal{S}_\alpha$:

1. If there exist $Z_n \neq 0$ and $Z_m \neq 0$, then the stabilizer of Z is $\{0\}$ and the orbit is equivalent to the line $\mathbf{R}$. These are the principal orbits.
2. The singular orbits, *i.e.* the non principal orbits, are the subspaces

$$S^1_n = \{Z \in S^\infty_\alpha \mid Z_m = 0 \text{ if } m \neq n\}, \quad \text{with} \quad n \in \mathbf{Z}.$$

Each singular orbit, equipped with the subset diffeology, is equivalent to the circle S^1. They are pure sounds made of only one harmonic.

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The Principal and Singular Loci

The singular locus of the action of the solenoid is

$$\mathcal{S}\mathrm{ing} = \bigcup_{n \in \mathbf{Z}} \mathrm{S}^1_n \subset \mathrm{S}^\infty_\alpha.$$

Equipped with the subset diffeology, it is the diffeological sum

$$\mathcal{S}\mathrm{ing} = \coprod_{n \in \mathbf{Z}} \mathrm{S}^1_n, \quad \text{and} \quad \dim(\mathcal{S}\mathrm{ing}) = 1.$$

It is a closed subset for the D-topolgy.

The regular or principal subspace, that is,

$$\mathcal{R}\mathrm{eg} = \mathrm{S}^\infty_\alpha - \bigcup_{n \in \mathbf{Z}} \mathrm{S}^1_n,$$

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THEOREM. There exists a closed 2-form ϖ on $\mathrm{QP}^\infty_\alpha$ such that:

$$\mathrm{pr}^*(\varpi) = \omega \restriction \mathrm{S}^\infty_\alpha.$$

NOTE 1. Because of the singular orbits, the quasi projective space is not transitive under the local automorphisms, and therefore ϖ is not symplectic. I say parasymplectic. It would not be surprising if the universal moment map would be injective. That can be checked later on.

NOTE 2. Considering the mechanism of the proof, it is clear that this situation is a special case of a more general theorem on reduction by $\mathbf{R}$ or S^1 actions.

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Symplectic Reduction — Proof i/v

Proof We shall apply the general criterion for a differential form to be a pullback by a subduction.

Let $\mathrm{P} : \mathrm{U} \to \mathrm{S}^\infty_\alpha$ and $\mathrm{P}' : \mathrm{U} \to \mathrm{S}^\infty_\alpha$ be two plots

$$\begin{array}{ccc} \mathrm{U} & \overset{\mathrm{P}}{\underset{\mathrm{P}'}{\rightrightarrows}} & \mathrm{S}^\infty_\alpha \\ & & \Big\downarrow \mathrm{pr} \\ & & \mathrm{QP}^\infty_\alpha \end{array} \qquad \text{such that} \qquad \mathrm{pr} \circ \mathrm{P} = \mathrm{pr} \circ \mathrm{P}'.$$

We want to check if, in these conditions, $\omega(\mathrm{P}) = \omega(\mathrm{P}')$?

That would insure the existence of ϖ, a (necessarily closed) 2-form on $\mathrm{QP}^\infty_\alpha$ such that $\omega = \mathrm{pr}^*(\varpi)$.

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We consider first of all what happens on the open subset

$$U_0 = P^{-1}(S^{\infty}_{\alpha} - \mathcal{S}\mathrm{ing}).$$

Since $S^{\infty}_{\alpha} - \mathcal{S}\mathrm{ing}$ is a union of orbits and $\mathrm{pr} \circ P = \mathrm{pr} \circ P'$, $P^{-1}(S^{\infty}_{\alpha} - \mathcal{S}\mathrm{ing}) = P'^{-1}(S^{\infty}_{\alpha} - \mathcal{S}\mathrm{ing}) = U_0$.

Now, the restrictions of P and P' on U_0 take their values in the subset of S^{∞}_{α} made of principal orbits of $\mathbf{R}$, for which the stabilizer of the action of $\mathbf{R}$ is $\{0\}$.

Thus, for each $r \in U_0$ there is a unique $\tau(r) \in \mathbf{R}$ such that, for all n,

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Symplectic Reduction — Proof ii/v

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The function τ is smooth. Indeed, for all $r_0 \in U_0$, there exists $n \in \mathbf{Z}$ such that $Z_n(r_0) \neq 0$.

Then there exists a neighborhood of r_0 where $Z_n(r) \neq 0$.

Hence, on this neighborhood:

$$e^{2i\pi\alpha_n\tau(r)} = \frac{Z_n'(r)}{Z_n(r)}.$$

But $r \mapsto Z_n'(r)$ and $r \mapsto Z_n(r)$ are smooth, thus $r \mapsto e^{2i\pi\alpha_n\tau(r)}$ is smooth, and therefore so is τ; because $\iota : \mathbf{R} \mapsto T^\infty$, such that $\mathcal{S}_\alpha = \iota(\mathbf{R})$, is an induction.

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Now, $\omega = d\varepsilon$, and

$$\begin{aligned}
\varepsilon(P')_r(\delta r) &= \frac{1}{2i\pi}\sum_{n\in\mathbf{Z}} \overline{Z'_n(r)}\,\frac{\partial Z'_n(r)}{\partial r}(\delta r) \\
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&= \varepsilon(P)_r(\delta r) + \tau^*(dt)_r(\delta r).
\end{aligned}$$

Therefore, since $d[\tau^*(dt)] = 0$, restricted to U_0, $d\varepsilon(P') = d\varepsilon(P)$.
That is, $[\omega(P') - \omega(P)] \restriction U_0 = 0$.

Symplectic Reduction — Proof iv/v

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Thus, by continuity, $[\omega(P') - \omega(P)] \upharpoonright \overline{U}_0 = 0$, where $\overline{U}_0$ is the closure of U_0. It remains to check what happens on the complementary $V = U - \overline{U}_0$.

The subset V is open, thus $P \upharpoonright V$ and $P' \upharpoonright V$ are two plots of S^∞_α with values in the subset of singular orbits $\mathcal{S}$ing.

But $\mathcal{S}$ing has dimension 1 and ω is a 2-form, thus: $\omega(P \upharpoonright V) = \omega(P' \upharpoonright V) = 0$. This is a general fact also true in diffeology.

In conclusion, $\omega(P') = \omega(P)$ everywhere on U. That proves that there exists a (closed) 2-form ϖ on $QP^\infty_\alpha = S^\infty_\alpha/\mathbf{R}$ such that $\mathrm{pr}^*(\varpi) = \omega$. $\square$

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Conclusion

In conclusion: the symplectic reduction of the Hamiltonian level of the action of the infinite solenoid on the infinite α-ellipsoid could be completed despite the infinite dimension of the spaces involved and the presence of singularities. This was made possible by the flexibility of the diffeology framework, without the need to introduce ad hoc heuristics. Only by using legitimate standard constructions of the diffeological framework.

Diffeology offers a unified framework for symplectic reduction with or without singularities in finite or infinite dimension.

This is what I wanted to show in this talk.

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The Actors of the Play

- $C^\infty_{\rm per}(\mathbf{R}, \mathbf{C})$ — The space of complex periodic functions.
- ω — The symplectic form on $C^\infty_{\rm per}(\mathbf{R}, \mathbf{C})$.
- $\mathcal{E}$ — The space of rapidly decreasing complex series.
- ω — The projection of ω on $\mathcal{E} \sim C^\infty_{\rm per}(\mathbf{R}, \mathbf{C})$.
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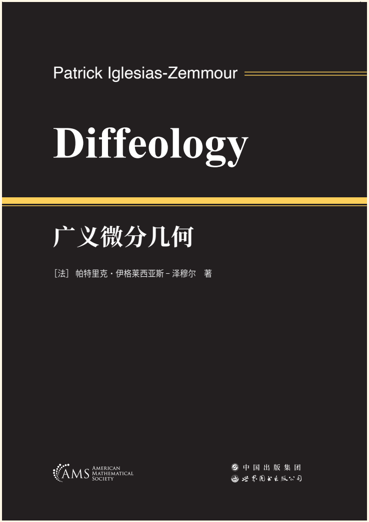

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Thank you!