Patrick Iglesias-Zemmour

Lectures on Diffeology

广义微分几何讲义

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הלל הזקן, משנה אבות א יד

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Preface

Since its creation, in the early 1980s, *diffeology* has become an alternative to —or rather an extension of— traditional differential geometry. With its developments in higher homotopy theory, fiber bundles, modeling spaces, Cartan-de Rham calculus, moment map and symplectic program, for examples, diffeology now covers a large spectrum of traditional fields and deploys them from singular quotients to infinite-dimensional spaces — and sometimes mixes the two — treating mathematical objects that are or are not strictly speaking manifolds, and other constructions, on an equal footing in a common framework.

We shall see some of its achievements through a series of examples, chosen because they are not covered by the geometry of manifolds, because they involve infinite-dimensional spaces or singular quotients, or both.

The growing interest in diffeology comes from the conjunction of two strong properties of the theory:

(1) First of all, the category {Diffeology} is stable under all set-theoretic constructions: sums, products, subsets, and quotients. One says that it is a complete and co-complete category. The space of smooth maps in diffeology has itself a natural *functional diffeology* for which the category is also Cartesian closed. This allows us to treat spaces of smooth maps between spaces as ordinary spaces. Mathematicians like these kind of categories for their stability with respect to set theoretic constructions.

(2) Just as important: quotient spaces, even heavily non-Hausdorff, Sikorski or Frölicher [KM97] etc. get a natural non-trivial and meaningful diffeology. This is particularly true of the *irrational tori*, where it all began, a non-Hausdorff quotients of the real line by a dense subgroup.[1] They own, as we shall see, a non trivial diffeology, capturing faithfully the intrication of the subgroup into its ambient space. This crucial property will be the raison d'être of many new constructions, or wide generalizations of classical constructions, that cannot exist in almost all other extensions of differential geometry.[2]

The handling of any kind of singularities, maybe more than the inclusion of infinite-dimensional spaces, reveals how diffeology changes the way we understand smoothness and discriminates this theory among the various alternatives. See, for example, the use of dimension in diffeology,[3] which distinguishes between the different half lines, quotients $\mathbf{R}^n/\mathrm{O}(n)$, on a pure differential geometry level.

In this regard, we can distinguish several types of singularities:

(a) A space can be singular when it presents a fatal mismatch between its topology and its diffeology, as in the case of the irrational torus.

(b) A point in a space can be singular when it behaves differently from other points under the action of diffeomorphisms. This is what happens in orbifolds, for example.

(c) A point in an embedding can behave differently from other points under the action of ambient diffeomorphisms. This is the case of the cusp in the semi-cubic.

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[1] More generally, quotient of Euclidean spaces $\mathbf{R}^n$ by generating dense and discrete (in the sense of diffeology) subgroups. They are also quotient of ordinary tori T^n by dense foliations.

[2] The kind of spaces that are trivial under the various generalizations of $\mathcal{C}^\infty$ differential geometry : I am not considering the various algebraic generalizations that do not play on the same level of intuition and generality and do not concern exactly the same sets/objects. We're indeed interested in differential geometry in the broadest sense.

[3] See [PIZ07].

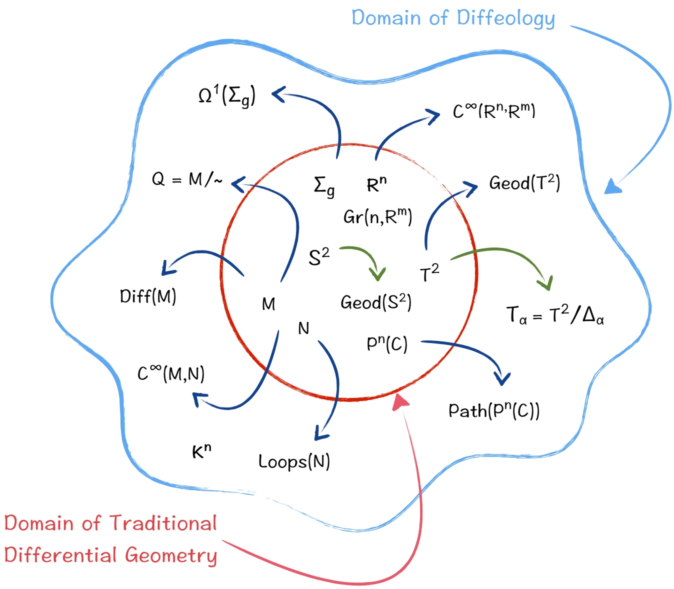

Figure 1. The scope of diffeology.

The Figure 1 shows the inclusivity of diffeology, with respect to differential constructions, in comparison with the classical theory.

The story began in the early 1980s, when Jean-Marie Souriau introduced his axiomatic called *difféologies* in a paper titled "Groupes Différentiels" [**Sou80**]. It was defined as a formal but light structure on groups,[4] and it was designed for dealing easily with infinite-dimensional groups of diffeomorphisms, in particular the group of symplectomorphisms or quantomorphisms. He named the groups equipped with such a structure *groupes différentiels*,[5] as announced in the title of his paper.[6] His definition was made of five axioms that we can decompose today into the first three that gave later the

[4]Compared to functional analysis heavy structures.

[5]Which translates in to English as "differential" or "differentiable groups."

[6]Actually, difféologies are built on the model of K.-T. Chen's *differentiable spaces* [**Che77**], for which the structure is defined over convex Euclidean subsets

notion of diffeology on arbitrary sets, and the last two, for the compatibility with the internal group multiplication. But it took three years, from 1980 to 1983, to separate the first three general axioms from the last two specific ones and to extract the general structure of *espace différentiel* from the definition of *groupe différentiel*. It is in particular the results on the *irrational torus* [DI83] and the development of homotopy [Igl85], that made urgent and unavoidable a formal separation between groups and spaces in the domain of Souriau's *differential structures*, as that gave a new spin to the theory as we know it today.[7]

Since its inception, the theory has evolved significantly and the book *Diffeology* is certainly a reference for the basic areas of the theory. I refer to it as the "textbook" in the following [TB].

This book, *Lectures on diffeology*, has not the same role, it is not a rewriting of the texbook despite having the same basis. The first half consists of notes from a series of lectures I gave at Shantou University in 2020/21, at the initiative of Enxin Wu, as part of a Chinese program inviting foreign professors. Writing down the lectures notes was part of the contract. In writing them, I've tried not to copy verbatim (although this is sometimes unavoidable) parts of the textbook, as it is more a support for these lectures. Instead, I've tried to extract the spirit of the constructions rather than their formalization. Readers can always refer to the relevant passage in the textbook for further details. I've tried to highlight the most important constructions that underpin the various branches of differential geometry, such as homotopy, fiber bundles, differential calculus and so on. And to show how they morph into diffeology, what makes them similar and what makes them different from what we are used to.

The second part of the book is a series of blog-notes I have posted on my website. These are various notes, remarks or exercises that I felt

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instead of open Euclidean domains. That makes diffeology more suitable to extending differential geometry than Chen's differentiable spaces, which focus more on homology and cohomology theories.

[7] I talk more on this story in the postface.

were interesting enough to share. I hope they will encourage some readers to develop the ideas or constructions they introduce. I am thinking in particular of Riemannian diffeology, which is still only a program. Symplectic diffeology, which is already well-developed, still needs more thought before it can become a acomplished theory. I have not been able to incorporate the recent developments and example of Čech cohomology that I have presented elsewhere, perhaps a future edition of these lectures will do justice to this issue.

And, this is because this book is more a work in progress than a finished thought, that I decided to present it as a series of typed notes, which explains it particular display and the choice of typography.

On an entirely different note, the special properties of diffeology: complete, co-complete and cartesian closed, have made this category a tool of choice for categoricians, especially in *differential homotopy* and *model category*.[8] Japan and China have got the lead in these two fields, that deserves to be mentioned. I asked Enxin Wu to contribute to this book on the subject. He gracefully accepted to write a short text which I have appended as a first step in this direction.

In conclusion, I will try to say why, from my point of view, diffeology is the perfect framework for differential geometry, to the point that it is exactly what we expect differential geometry to be. First of all, the geometer will find pleasant and useful the flexibility of diffeology, to extend in a unique formal and versatile framework, different constructions in various fields, without inventing each time a heuristic framework that momentarily satisfies its needs.

Then, beyond all these circumstances and technicalities, what does diffeology have to offer on a more formal or conceptual level? The answer lies partly in Felix Klein's Erlangen program [**Kle72**] :

> The totality of all these transformations we designate as the *principal group* of space-transformations;

[8] I cite just a couple of names on these subfields in rapid development, for example [**CW14**] in model category, and [**SYH18, Kih19, Kur20**] in differential homotopy.

> *geometric properties are not changed by the transformations of the principal group.* And, conversely, *geometric properties are characterized by their remaining invariant under the transformations of the principal group...*
>
> As a generalization of geometry arises then the following comprehensive problem:
>
> *Given a manifoldness and a group of transformations of the same; to investigate the configurations belonging to the manifoldness with regard to such properties as are not altered by the transformations of the group.*

As we know, these considerations are today regarded by mathematicians as the modern interpretation of the word/idea of *geometry*. A geometry is given as soon as a space and a group of transformations of this space are given.[9].

Consider, for example, Euclidean geometry, defined by the group of Euclidean transformations, our *principal group*, on the Euclidean space. We can interpret, for example, the *Euclidean distance* as the invariant associated with the action of the Euclidean group on the set of pairs of points. We can superpose a pair of points onto another pair of points, by an Euclidean transformation, if and only if the distance between the points is the same for the two pairs.[10] Hence, *geometric properties* or *geometric invariants* can be regarded as the orbits of the *principal group* in some spaces built on top of the principal space, and also as fixed/invariant points, since an orbit is a fixed point in the set of all the subsets of that space. In brief, what emerges from these considerations suggested by Felix Klein's principle is the following:

[9] Jean-Marie Souriau reduces the concept of geometry to the group itself [**Sou03**] But this is an extreme point of view I am not confident to share, for several reasons.

[10] We could continue with the case of triangles and other elementary constructions – circles, parallels etc. – involving the nature of stabilizers. We can compare between Euclidean and symplectic geometry, for example, from a strict Kleinian point of view. See the discussion in [**PIZ02**].

Claim. A geometry is associated with/defined by a *principal group* of transformations of some space, according to Klein's statement. The various geometric properties/invariants are described by the various actions of the principal group on spaces built on top of the principal space: products, sets of subsets, and so on. Each one of these properties, embodied as orbits, stabilizers, quotients, and so on, captures a part of this geometry.

Now, how does diffeology fit to this context?

∗ One can regard a diffeological space as the collection of the plots that gives its structure. This is the *passive approach*.

∗ Or we can look at the space through the action of its group of diffeomorphisms:[11] on itself, but also on its powers or parts or maps. This is the *active approach*.

This dichotomy appears already for manifolds, where the change of coordinates (transition functions of an atlas) is the passive approach. The active approach, as the action of the group of diffeomorphisms, is often neglected, and there are a few reasons for that. Among them, the group of diffeomorphisms is not a Lie group *stricto sensu* – it does not fit in the category {Manifolds} – and that creates a psychological issue. A second reason is that its action on the manifold itself is transitive,[12] there are no immediate invariants, one having first to consider some secondary/subordinate spaces to make the first invariants appear.

These obstacles, psychological or real, vanish in diffeology. First of all, the group of diffeomorphisms is naturally a diffeological group. And the square, discussed in one of the lectures, is an example of a space where the action of the group of diffeomorphisms, the main group in Klein's sense, captures a good first image of its geometry.[13]

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[11]We consider more precisely the action of the pseudo-group of local diffeomorphisms.

[12]Generally, manifolds are regarded as connected, Haussdorf, and second countable.

[13]And not just of its topology.

Indeed, it has three orbits: the corners, the edges and the interior. Any diffeomorphism preserves separately the interior of the square and its border, which is a consequence of the D-topology.[14] But the fact that a diffeomorphism of the square cannot map a corner into an edge is a typical smooth property.

Klein's principle can already be tested on this simple example. As the square is naturally an object of theory, there's no need for an heuristic extension here.

Claim. Considering the group of diffeomorphisms of a diffeological space as its principal group, we can look at diffeology as the formal framework that makes *differential geometry*, the geometry — in the sense of Felix Klein — of the group of diffeomorphisms. Or possibly, the (larger) pseudogroup of local diffeomorphisms.[15]

Actually, the action of local diffeomorphisms defines a stratification, called *Klein stratification*, that embodies the internal geometry of the object itself. The general geometry to which it belongs is defined by the specific category of objects to which it is formelly attached. For example, Euclidean geometry becomes the category of Euclidean objects, that is, the category of spaces equipped with an action of the Euclidean group with morphism equivariant maps.

Because diffeology is such a wide and stable category that satisfactorily encompasses so many diverse situations, from singular quotients to infinite dimensions, even mixing cases,[16] I think it is fair to say that it largely fulfils its mission and, according to the Kleinian viewpoint, answers the question once posed by a student: "In what way is diffeology geometry?"

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[14] A diffeomorphism is in particular an homeomorphism for the D-topology. This is an opportune moment to emphasise this: the diffeology approach is that topology is a consequence of differential structure, and not the other way round as it is usually the case, where the differential structure is subordinate to the topology.

[15] In comparison with [**Sou03**].

[16] See [**PIZ16**], for example.

Acknowledgements

My sincere thanks to Enxin Wu for he invited me to give a series of lectures on diffeology at Shantou University, as part of a “Guangdong Science and Technology Project”.[1] This led to the publication of these lecture notes, giving me the opportunity to present in China this new look at differential geometry, and to show some examples of applications.

It’s always a pleasure for me to thank the Hebrew University of Jerusalem Israel, to which I belong by heart. I am thinking to professor Elon Lindenstrauss and professor Jay Fineberg in particular, who a few years ago, as Chairman of the Mathematics Department and Dean of the Faculty of Science at the time, enabled me to continue my academic activity as a visiting professor, despite my retirement from the CNRS in France. For that, I am deeply grateful. This is thanks to them and to all my colleagues at HUJI, Emanuel Farjoun, Itamar Cwik, Yael Karshon, Jake Solomon, to cite only a few, and all those who have always welcomed me, that this book could exist.

I also would like to take this opportunity to thank my Japanese colleagues, in particular Norio Iwase and Katsuhiko Kuribayashi, for their many invitations to take part in the conferences on differential

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homotopy, where diffeology is also abundantly discussed, that they have been organizing regularly in Japan for several years. They are a source of inspiration and questioning, as well as prospects for further development. I hope that these notes will be useful to their students, some were inspired by discussions I have had with them.

And lastly, I'd like to express my sincere gratitude and thanks, for the second time, to all the members of the Beijing WPC team who helped me so much, in particular to the publishing director Liang Chen, and especially to Yeqing Liu, my ever-patient editor, who made this publication possible and put the finishing touches to it.

Patrick Iglesias-Zemmour
Einstein Institute of Mathematics
Edmond J. Safra Campus
The Hebrew University of Jerusalem
Givat Ram. Jerusalem, 9190401, Israel

http://math.huji.ac.il/~piz
piz@math.huji.ac.il

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At the Beginning...

This lecture presents the prerequisites for the study of diffeology. As we shall see, a good understanding of indefinitely differentiable maps on Euclidean domains, and a few fundamental theorems of calculus, will be a good start.

Let us introduce a little bit of vocabulary. First of all, we call *Euclidean space* any real vector space $\mathbf{R}^n$ for some n.

The Euclidean structure of $\mathbf{R}^n$, defined by the inner product

$$x \cdot y = \sum_{i=1}^{n} x^i y^i \quad \text{with} \quad \|x\| = \sqrt{\sum_{i=1}^{n} (x^i)^2},$$

is used to define its (standard) topology: an open subset $\mathcal{O}$ in $\mathbf{R}^n$ is any *union of open balls*, like:

$$\mathcal{O} = \bigcup_{i \in \mathcal{I}} \mathcal{B}(r_i, \varepsilon_i),$$

where $\mathcal{I}$ is any set of indices, r_i is any point in $\mathbf{R}^n$ and ε_i is any (strictly) positive number.

Therefore, a subset $\mathrm{U} \in \mathbf{R}^n$ is open for the standard topology if (and only if): for each $r \in \mathrm{U}$ there exists an open ball $\mathcal{B}(r, \varepsilon)$ included in U, $\mathcal{B}(r, \varepsilon) \subset \mathrm{U}$.

Next, we call *Euclidean domain* any open subset of an Euclidean space. We generally denote them by some big letter U, V, W etc. and also by cursive letters $\mathcal{O}$ etc.

We denote also by Domains($\mathbf{R}^n$) the set of all Euclidean domains in $\mathbf{R}^n$. We call them simply *n-domains*.

Note that this is just the topology of $\mathbf{R}^n$ sometimes denoted by $\{\mathrm{Top}\}(\mathbf{R}^n)$.

A domain in $\mathbf{R}^n$ is said to be of dimension n, or be an n-domain.

Now, about *continuity*. A map $F: U \to V$, where U and V are Euclidean domains, is said to be *continuous* if (and only if): the pullback $f^{-1}(\mathcal{O})$ of any open subset $\mathcal{O} \subset V$ is an open subset in U.

That is equivalent to say that for any open ball $\mathcal{B} \subset V$, there exists a family of open balls $\mathcal{B}_i \subset U$ such that $f^{-1}(\mathcal{B}) = \cup_i \mathcal{B}_i$, for some family of indices $\mathcal{I}$.

1. Differentiable and smooth paths

Consider a *path*

$$\gamma \colon]a, b[\to \mathbf{R}^n.$$

Assume that γ is continuous. Then, consider the map

$$\Delta\gamma \colon (t, t') \mapsto \frac{\gamma(t') - \gamma(t)}{t' - t},$$

defined on

$$]a, b[^2 - \{(t, t)\}_{a<t<b},$$

with values in $\mathbf{R}^n$. The closed subset $\{(t, t)\}_{a<t<b}$ is called the *diagonal*. The function $\Delta\gamma$ is continuous, but:

1. Definition. *If $\Delta\gamma$ extends continuously on the diagonal, we say that γ is continuously differentiable or of class $\mathcal{C}^1$.*

We denote by

$$\dot\gamma(t) \quad \text{or} \quad \frac{d\gamma(t)}{dt} \quad \text{the value} \quad \Delta\gamma(t, t) = \lim_{t' \to t} \Delta\gamma(t', t).$$

The function $\dot\gamma :]a,b[\to \mathbf{R}^n$ is then called the *derivative* of γ. The value $\dot\gamma(t)$ is the derivative of γ at the point t, or *at time* t.

Now, if γ is of class $\mathcal{C}^1$, then $\dot\gamma$, which is defined on the same interval, is continuous. One can ask if $\dot\gamma$ is also $\mathcal{C}^1$? If it is the case, we generally denote by $\ddot\gamma$ the derivative of $\dot\gamma$. We say that γ is of class $\mathcal{C}^2$.

That leads to the definition:

2. Definition. *We say that γ is of class $\mathcal{C}^k$, $k > 1$, if (and only if):*

$$\gamma \textit{ is of class } \mathcal{C}^{k-1} \textit{ and } \frac{d^{k-1}\gamma(t)}{dt^{k-1}} \textit{ is of class } \mathcal{C}^1.$$

And we say that γ is smooth, or of class $\mathcal{C}^\infty$, if γ is of class $\mathcal{C}^k$ for all $k \in \mathrm{N}$. Note that continuous paths are said to be of class $\mathcal{C}^0$.

2. Smooth maps, the holistic approach

In diffeology we are interested principally in *smooth maps*, or *infinitely differentiable maps*. Precisely, diffeology consists in extending the concept of smooth maps from Euclidean domains to arbitrary sets. That is why we have the choice in the definitions of smooth maps between Euclidean domains. We will present them here, not in their historic order but according with what we think is more speaking to our mind. The first definition is actually a theorem.

3. Definition-Theorem. (Boman 1967) *Let $\mathrm{F} : \mathrm{U} \to \mathrm{V}$ be a continuous map between two Euclidean domains. The map F is infinitely differentiable, or smooth, if (and only if) the composite $\mathrm{F}\circ\gamma$, where γ is any smooth path in U, is a smooth path in V.*

As we have seen, smooth paths have been defined previously, independently.

3. Differentiable maps, the pedestrian approach

Now, the classical definition of differentiable map begins with the definition of *tangent maps*.

4. Definition. *Let f and g be two maps defined on a same Euclidean domain U to some $\mathbf{R}^m$. Let r_0 be some point in U. We say that f and g are tangent in r_0 if:*

$$\lim_{r \to r_0} \frac{f(r) - g(r)}{\|r - r_0\|} = 0.$$

Of course, that implies in particular that $f(r_0) = g(r_0)$.

5. Definition. *Let f be a map defined on some domain $\mathrm{U} \in \mathbf{R}^n$ with values in $\mathbf{R}^m$. We say that f is differentiable, or derivable,[1] at the point $r_0 \in \mathrm{U}$ if it is tangent, at this point, to an* affine map

$$r \mapsto f(r_0) + \mathrm{M}(r - r_0),$$

where $\mathrm{M} \in \mathrm{L}(\mathbf{R}^n, \mathbf{R}^m)$ is a linear map from $\mathbf{R}^n$ to $\mathbf{R}^m$. That condition means precisely that,

$$\lim_{r \to r_0} \frac{f(r) - f(r_0) - \mathrm{M}(r - r_0)}{\|r - r_0\|} = 0. \qquad (\clubsuit)$$

We say that f is differentiable on U, *or simply differentiable, without more precision, if f is tangent to an affine map everywhere.*

6. Remark. The condition ($\clubsuit$) implies that $f(r)$ converges to $f(r_0)$ when r tends to r_0. That is f is continuous in r_0: to be tangent to an affine map implies to be continuous: *a differentiable map is necessarily continuous.*

Proof. Since $\lim_{r \to r_0} \frac{f(r) - f(r_0) - \mathrm{M}(r - r_0)}{\|r - r_0\|} = 0$, $\lim_{r \to r_0} f(r) - f(r_0) - \mathrm{M}(r - r_0) = 0$. Thus, since $\lim_{r \to r_0} \mathrm{M}(r - r_0) = 0$, $\lim_{r \to r_0} f(r) - f(r_0) = 0$. That is $f(r) \xrightarrow[r \to r_0]{} f(r_0)$, and f is continuous at r_0. That is the logic of the situation. More formally, since $\lim_{r \to r_0} f(r) - f(r_0) - \mathrm{M}(r - r_0) = 0$, for all $\varepsilon > 0$, there exists $\eta > 0$ such that $\|r - r_0\| < \eta$ implies $\|f(r) - f(r_0) - \mathrm{M}(r - r_0)\| < \varepsilon$. On the other hand, let m be $\|\mathrm{M}\| = \sup_{\|u\|=1} \|\mathrm{M}(u)\|$ (remember, the sphere S^{n-1} is compact). Thus, for all r, $\|\mathrm{M}(r - r_0)\| \leq m\|r - r_0\|$. That is, for all $\varepsilon > 0$ there is $\eta' > 0$ such that $\|r - r_0\| < \eta$ implies $\|\mathrm{M}(r - r_0)\| < \varepsilon$, actually η' can be chosen equal to ε/m. Now, the inequality on a triangle $u + v = w$ says that $\|w\| \leq \|u\| + \|v\|$, or $\|w\| - \|u\| \leq \|v\|$. Applied to

[1] The words *differentiable* and *derivable* are here completely equivalent.

$u = M(r-r_0)$, $w = f(r)-f(r_0)$ and $v = w-u = f(r)-f(r_0)-M(r-r_0)$, that gives: $\|f(r)-f(r_0)\| - \|M(r-r_0)\| \le \|f(r)-f(r_0)-M(r-r_0)\| \le \varepsilon$. Then, $\|f(r)-f(r_0)\| \le \varepsilon + \|M(r-r_0)\|$. Now, chosing $\eta'' < \inf(\eta, \eta')$, we get $\|f(r)-f(r_0)\| \le \varepsilon + \varepsilon$. Changing ε to $\varepsilon/2$, we get: for all $\varepsilon > 0$, there exists $\eta'' > 0$ such that $\|r - r_0\| < \eta''$ implies $\|f(r)-f(r_0)\| \le \varepsilon$. Therefore, f is continuous at r_0. ▶

Now let us describe more precisely the linear part of the affine tangent map. Pick a vector $v \in \mathbf{R}^n$, $v \neq 0$, and let $r = r_0 + tv$, where $t > 0$ is small enough for r to be in the domain U. The last condition writes:

$$\lim_{t\to 0} \frac{f(r_0+tv) - f(r_0) - M(tv)}{t\|v\|} = 0.$$

And then,

$$M(v) = \lim_{t\to 0} \frac{f(r_0+tv) - f(r_0)}{t}.$$

The linear map M is then called the *tangent linear map* and it is denoted by

$$Df_{r_0}(v) \text{ or } Df(r_0)(v) \text{ or } D(f)(r_0)(v) = \lim_{t\to 0} \frac{f(r_0+tv) - f(r_0)}{t}.$$

So, f is differentiable on U if it admits a tangent linear map at every point in U,

$$f' \text{ or } Df \text{ or } D(F) \colon U \to L(\mathbf{R}^n, \mathbf{R}^m).$$

The *affine tangent map* at the point r_0 writes then

$$r \mapsto f(r_0) + D(f)(r_0)(r - r_0).$$

Note that there is another way to express the approximation of the function f by its affine tangent map around r_0:

$$f(r) = f(r_0) + D(f)(r_0)(r - r_0) + o(\|r - r_0\|),$$

where Landau's *Little-O* notation $o(x)$ means $\lim_{x\to 0} o(x)/x = 0$.

As we know the set of linear maps $L(\mathbf{R}^n, \mathbf{R}^m)$ is a real vector space of dimension $n \times m$, equivalent to $\mathbf{R}^{n\times m}$, as such:

7. Definition. *We say that $f \colon U \to \mathbf{R}^m$ is of class $\mathcal{C}^1$ if f is differentiable on U, and if the map that associates its tangent linear map* *with each point in* U *is continuous. That is, if the map* $r \mapsto Df(r)$, *defined on* U *with values in* $L(\mathbf{R}^n, \mathbf{R}^m)$ *is continuous.*

8. Remark. An affine map from $\mathbf{R}^n$ to $\mathbf{R}^m$ writes $x \mapsto Ax + b$, where $A \in L(\mathbf{R}^n, \mathbf{R}^m)$ and $b \in \mathbf{R}^m$. The set of these affine maps is denoted by $\mathrm{Aff}(\mathbf{R}^n, \mathbf{R}^m)$, it is naturally a vector space of dimension $m + n \times m$, equivalent to the space of matrices

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \quad \text{such that} \quad \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + b \\ 1 \end{pmatrix},$$

and inherits the standard topology of $\mathbf{R}^{m+n\times m}$. The affine tangent map writes, for all $r \in U$:

$$\begin{pmatrix} Df_r & f(r) - Df_r(r) \\ 0 & 1 \end{pmatrix}.$$

We can notice that, to be $\mathcal{C}^1$, it is equivalent to request that the maps that associate the linear tangent map in $L(\mathbf{R}^n, \mathbf{R}^m)$, or the affine tangent map in $\mathrm{Aff}(\mathbf{R}^n, \mathbf{R}^m)$, with every point r in U, are continuous. However, on a conceptual level: to be differentiable implies to be continuous, then, having a *first local affine approximation*, — which is exactly what the affine tangent map is — continuous everywhere is a natural request. And that is the meaning of being $\mathcal{C}^1$.

4. Higher order derivatives and smooth maps

Now, we can look for *higher order derivatives*. Let us begin by the second order. Assume that f is derivable, then its derivative $D(F)$ is defined on U with values in $L(\mathbf{R}^n, \mathbf{R}^m)$. The second derivative of f is then defined by

$$D^2(f)(r) = D\big(r \mapsto D(f)(r)\big)(r) \in L\big(\mathbf{R}^n, L(\mathbf{R}^n, \mathbf{R}^m)\big).$$

Note that if f is derivable and if its derivative Df is derivable, that implies that Df is continuous and f is $\mathcal{C}^1$.

So, we get a recursive definiton of higher derivative of f and then a recursive definition of class $\mathcal{C}^k$:

9. Definition. *The kth-derivative of f, if it exists, is the derivative denoted and defined recursively by*

$$\mathrm{D}^k(f) = \mathrm{D}\big(r \mapsto \mathrm{D}^{k-1}(f)(r)\big).$$

Note that, if f is differentiable until order k, then $\mathrm{D}^\ell(f)$ is continuous until $\ell = k-1$, thanks to Remark 6. Therefore, if f admits a kth-derivative, then f is $\mathcal{C}^{k-1}$, with the definition:

10. Definition. , *A* function *f is* $\mathcal{C}^k$*, or of* class $\mathcal{C}^k$*, if f is derivable till order k and its k-derivative is continuous.*

We say that f is infinitely differentiable*, or* infinitely derivable*, or* smooth*, if f is of class $\mathcal{C}^k$ for all integer k.*

11. Remark. Boman theorem cited previously says exactly that definition 9 and Definition-Theorem 3 are coherent. Beware, it is a subtle and non-trivial theorem.

5. The tangent linear map

We consider a differentiable map $f\colon \mathrm{U} \to \mathbf{R}^m$, with $\mathrm{U} \subset \mathbf{R}^n$ an open subset.

Let us denote by x and y the source and target variables involved in f, that is, $f\colon x \mapsto y$ with $x = (x_1, \ldots, x_n) \in \mathrm{U}$ and $y = (y_1, \ldots, y_m) \in \mathbf{R}^m$. The tangent linear map can be written indifferently

$$\mathrm{D}(f)(x) \text{ or } \mathrm{D}(x \mapsto y)(x),$$

depending what we want to focus on. For example, if you denote the square root function by sqrt, you can write it derivative as D(sqrt)(x), or $\mathrm{D}(x \mapsto \sqrt{x})(x)$.

Now, decompose x, v and y on the canonical basis,

$$x = \sum_{i=1}^{n} x^i \boldsymbol{e}_i, \quad v = \sum_{j=1}^{m} v^i \boldsymbol{e}_i \quad \text{and} \quad y = \sum_{j=1}^{m} y^j \boldsymbol{e}_j \quad \text{with} \quad y^j = f^j(x).$$

Now, the tangent linear map writes

$$\mathrm{D}(x \mapsto y)(x)(v) \quad = \quad \mathrm{D}(x \mapsto y)(x)\left(\sum_{i=1}^{n} v^i \boldsymbol{e}_i\right)$$

$$\begin{aligned}
&= \sum_{i=1}^{n} v^i \mathrm{D}(x \mapsto y)(x)(\boldsymbol{e}_i) \\
&= \sum_{i=1}^{n} v^i \mathrm{D}\left(x \mapsto \sum_{j=1}^{m} y^j \boldsymbol{e}_j\right)(x)(\boldsymbol{e}_i) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} v^i \mathrm{D}\left(x \mapsto y^j\right)(x)(\boldsymbol{e}_i)\cdot \boldsymbol{e}_j
\end{aligned}$$

In other words, the tangent linear map $\mathrm{D}(x \mapsto y)(x)$ is represented by the matrix $(\mathrm{D}_i^j)_{i=1}^{n}{}_{j=1}^{m}$, where

$$\mathrm{D}_i^j = \mathrm{D}\left(x \mapsto y^j\right)(x)(\boldsymbol{e}_i).$$

Now, what does exactly represent $\mathrm{D}\left(x \mapsto y^j\right)(x)(\boldsymbol{e}_i)$?

$$\mathrm{D}\left(x \mapsto y^j\right)(x)(\boldsymbol{e}_i) = \lim_{t\to 0} \frac{f^j(x_1,\dots,x_i+t,\dots,x_n) - f^j(x_1,\dots,x_i,\dots,x_n)}{t},$$

which is, by definition, the partial derivative

$$\mathrm{D}_i^j = \frac{\partial y^j}{\partial x^i} \quad \text{also denoted by} \quad \partial_i y^j.$$

<u>Notations:</u> By commodity we will denote also

$\mathrm{D}(x \mapsto y)(x)$ by $\frac{\partial y}{\partial x}$, the derivative;

and

$\mathrm{D}(x \mapsto y)(a)$ by $\left.\frac{\partial y}{\partial x}\right|_{x=a}$, the value of the derivative.

Eventually, the tangent linear map can by written as the matrix of partial derivatives

$$\mathrm{D}(x \mapsto y)(x) = \mathrm{D}\left(\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} \mapsto \begin{pmatrix} y^1 \\ \vdots \\ y^m \end{pmatrix}\right)(x) = \begin{pmatrix} \partial_1 y^1 & \cdots & \partial_n y^1 \\ \vdots & \ddots & \vdots \\ \partial_1 y^m & \cdots & \partial_n y^m \end{pmatrix}$$

such that

$$\mathrm{D}(x \mapsto y)(x)(v) = \begin{pmatrix} \partial_1 y^1 & \cdots & \partial_n y^1 \\ \vdots & \ddots & \vdots \\ \partial_1 y^m & \cdots & \partial_n y^m \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n v^i \partial_i y^1 \\ \vdots \\ \sum_{i=1}^n v^i \partial_i y^m \end{pmatrix}.$$

12. Proposition. *Let $f\colon \mathrm{U} \to \mathrm{V}$ and $g\colon \mathrm{V} \to \mathrm{W}$ be two $\mathcal{C}^1$ maps, with $\mathrm{U} \subset \mathbf{R}^n$, $\mathrm{V} \subset \mathbf{R}^m$ and $\mathrm{W} \subset \mathbf{R}^\ell$. The derivative of the composite, which is also $\mathcal{C}^1$, satisfies the so-called* chain-rule*:*

$$\mathrm{D}(g \circ f)(r) = \mathrm{D}(g)(f(r)) \circ \mathrm{D}(f)(r).$$

Note also that the derivation D *is linear. If f_1 and f_2 are defined from* U *to* V *and $\lambda, \mu \in \mathbf{R}$, then:*

$$\mathrm{D}(\lambda f_1 + \mu f_2)) = \lambda \mathrm{D}(f_1) + \mu \mathrm{D}(f_2).$$

13. Note. The chain-rule writes in term of partial derivatives:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}\frac{\partial y}{\partial x},$$

and the value at the point $x = a$, with where $b = f(a)$:

$$\left.\frac{\partial z}{\partial x}\right|_{x=a} = \left.\frac{\partial z}{\partial y}\right|_{y=b} \left.\frac{\partial y}{\partial x}\right|_{x=a}.$$

In terms of coordinates:

$$\partial_i z^j = \sum_{k=1}^{m} \partial_i y^k \partial_k z^j \quad \text{that is} \quad \frac{\partial z^j}{\partial x^i} = \sum_{k=1}^{m} \frac{\partial z^j}{\partial y^k}\frac{\partial y^k}{\partial x^i},$$

where $i = 1 \ldots n$, $k = 1 \ldots m$ and $j = 1 \ldots \ell$.

14. Note. In particular, for $\gamma\colon \mathcal{I} \to \mathrm{U}$, a smooth path in U,

$$\mathrm{D}(f \circ \gamma)(t)(1) = \mathrm{D}(f)(\gamma(t)) \circ \mathrm{D}(\gamma)(t)(1),$$

but

$$\mathrm{D}(\gamma)(t)(1) = \lim_{\varepsilon \to 0} \frac{\gamma(t + \varepsilon \times 1) - \gamma(t)}{\varepsilon} = \frac{d\gamma(t)}{dt} = \dot{\gamma}(t).$$

Therefore:

$$\mathrm{D}(f \circ \gamma)(t)(1) = \mathrm{D}(f)(\gamma(t))\big(\dot{\gamma}(t)\big).$$

15. Notation. It hapens that a tangent vector $v \in \mathbf{R}^n$ would be denoted by a variational notation δx, meaning that v is regarded as a *variation* of the point x. Write $\gamma\colon t \mapsto x$ and $f\colon x \mapsto y$, then

$f \circ \gamma \colon t \mapsto y$, meaning $t \mapsto x \mapsto y$. Using the partial derivative notation introduced previously, the formula above writes:

$$\delta y = \frac{\partial y}{\partial x}(\delta x),$$

with

$$x = \gamma(t),\ \ y = f(x),\ \ \delta x = \frac{dx}{dt},\ \ \delta y = \frac{dy}{dt},$$

to which we can add:

$$\delta x = \frac{\partial x}{\partial t}(\delta t) \quad \text{and} \quad \delta t = 1 \ \Rightarrow\ \delta x = \frac{dx}{dt}.$$

Of course, every vector v can be realised as a variation of the point x by considering $\gamma(t) = x + tv$.

6. Higher derivatives components

What have been made for the tangent linear map can be done for higher derivatives. For example, let $f \colon x \mapsto y$. The second derivative

$$\mathrm{D}^2(f)(x) = \mathrm{D}\big(x \mapsto \mathrm{D}(f)(x)\big)(x)$$

is represented itself by the bilinear form:

$$\mathrm{D}^2(f)(x)(v)(w) = \mathrm{D}\big(x \mapsto \mathrm{D}(f)(x)(v)\big)(x)(w),$$

for all $v, w \in \mathbf{R}^n$. Let decompose $v = \sum_{i=1}^n v^i \boldsymbol{e}_i$ and $w = \sum_{j=1}^n w^j \boldsymbol{e}_j$. We get:

$$\begin{aligned}
\mathrm{D}^2(f)(x)(v)(w) &= \mathrm{D}\Big(x \mapsto \mathrm{D}(f)(x)\Big(\sum_{i=1}^n v^i \boldsymbol{e}_i\Big)\Big)(x)\Big(\sum_{j=1}^n w^j \boldsymbol{e}_j\Big) \\
&= \sum_{j=1}^n w^j \mathrm{D}\big(x \mapsto \sum_{i=1}^n v^i \mathrm{D}(f)(x)(\boldsymbol{e}_i)\big)(x)(\boldsymbol{e}_j) \\
&= \sum_{j=1}^n w^j \sum_{i=1}^n v^i \mathrm{D}\big(x \mapsto \mathrm{D}(f)(x)(\boldsymbol{e}_i)\big)(x)(\boldsymbol{e}_j) \\
&= \sum_{j,i=1}^n w^j v^i \frac{\partial}{\partial x^j}\left(\frac{\partial y}{\partial x^i}\right).
\end{aligned}$$

And since $y = (y^1, \ldots, y^m)$,

$$\mathrm{D}^2(f)(x)(v)(w) = \sum_{j,i=1}^{n} w^j v^i \begin{pmatrix} \frac{\partial^2 y^1}{\partial x^j \partial x^i} \\ \vdots \\ \frac{\partial^2 y^m}{\partial x^j \partial x^i} \end{pmatrix} = \sum_{j,i=1}^{n} w^j v^i \begin{pmatrix} \partial^2_{ji} y^1 \\ \vdots \\ \partial^2_{ji} y^m \end{pmatrix}$$

The partial derivatives $\partial_{ij} y^k$ are the components of the bilinear map $\mathrm{D}^2(f)(x)$.

A main property of continuously differentiable function is the commutativity of the partial derivatives:

16. Theorem (Schwarz). *Let $f\colon \mathrm{U} \to \mathbf{R}^m$, with U a n-domain, be a $\mathcal{C}^2$ map. Then, the second derivative $\mathrm{D}^2(f)(x)$ is symetric. In other words, the partial derivatives commutent:*

$$\frac{\partial}{\partial x^i}\left(\frac{\partial y^k}{\partial x^j}\right) = \frac{\partial}{\partial x^j}\left(\frac{\partial y^k}{\partial x^i}\right) \quad \textit{i.e.} \quad \partial^2_{ij} y^k = \partial^2_{ji} y^k.$$

For $\mathcal{C}^k$ maps, the k-derivatives $\mathrm{D}^k(f)(x)$ is a k-linear map, with components the partial derivatives:

$$\partial^k_{i_1 \ldots i_k} y^\ell = \partial_{i_1}\left(\partial^{k-1}_{i_2 \ldots i_k} y^\ell\right)$$

And, thanks to the last proposition, this multilinear map is symetric: the order of the indices does not matter.

17. Theorem. *Let $f\colon \mathrm{U} \to \mathbf{R}^m$ be a map, where U is an n-domain. The map f is smooth if and only if f admits partial derivatives at any order.*

☞ Proof. Indeed, and that is the main point: if all the partial derivatives at any order exist, that implies in particular that they are continuous, thanks to § 6, and then the map f is *continuously infinitely differentiable.* ▶

This theorem is a practical criterion to check if a map is smooth.

That is everything we need to know to introduce and study diffeology. That is what makes diffeology a good alternative to the usual teaching of differential geometry. And we shall see in the future how it could be understood formally as a *geometry in the sense of Felix Klein.*

7. The category of Euclidean domains

Euclidean domains are the objects of a category for which the arrows are the smooth maps. We denote it by {Euclidean Domains}.

The goal of diffeology is to transfer some properties of the category of Euclidean domains to arbitrary sets.

8. Some theorems we should know

There are a few important theorems of differential calculus in $\mathbf{R}^n$ we will need in future development of diffeology, for example the implicit function theorem or the rectification of vector fields and some others. However, they are not necessary for now to learn diffeology. That is why it is better to introduce them only when they will be used.

Notes

Diffeology, the Axiomatic

In this lecture we will look at the short axiomatic that founds diffeology. It describes the basic set theoretic constructions of the theory and give a few examples.

Diffeologies are defined on arbitrary sets without any preexisting structure, neither topology nor anything else. That is important enough to be underlined and remembered. Diffeology is based on the notion of parametrizations, and will consists in declaring which parameterizations in a set will be regarded as smooth, provided that a small set of axioms is satisfied. Then, the development of diffeology will consist in transferring, through these specific parametrizations, significative properties and constructions a priori defined in the category of smooth domains, such as homotopy groups, fiber bundles, differential form. . . for examples.

9. What is a diffeology?

The theory of diffeology begins with the idea of parametrization. The first step in this direction was taken by K.T. Chen in his paper on "Iterated Path Integrals" [**Che77**], but the parametrizations were defined on convex subsets of Euclidean domains. In 1980, in his paper on "Groupes différentiels" [**Sou80**], J.M. Souriau keeps the same axiomatic but with parametrizations defined on Euclidean domains, open subsets of Euclidean spaces. That is what founds today's diffeology:

18. Parametrizations. We call *parameterization* in a set X any map $P: U \to X$ where U is some Euclidean domain, that is, any open subset of an Euclidean space. If we want to be specific, we say that P is an *n-parameterization* when U is an open subset of $\mathbf{R}^n$. The set of all parameterizations in X is denoted by

$$\mathrm{Param}(X) = \{P: U \to X \mid U \in \mathrm{Domains}(\mathbf{R}^n), n \in \mathbf{R}\}$$

Note that there is no condition of injectivity on P, and as we said, neither any topology precondition on X a priori.

19. Diffeology. A *diffeology* on a set X is any subset

$$\mathcal{D} \subset \mathrm{Param}(X)$$

that satisfies the following axioms:

1. *Covering*: $\mathcal{D}$ contains the constant parameterizations.
2. *Locality*: Let $P: U \to X$ be parametrization. If, for all $r \in U$, there is an open neighbourhood V of r such that $P \restriction V \in \mathcal{D}$, then $P \in \mathcal{D}$.
3. *Smooth compatibility*: For all $P: U \to X$ in $\mathcal{D}$, for all $F \in \mathcal{C}^\infty(V, U)$, where V is an Euclidean domain, $P \circ F \in \mathcal{D}$.

A space equipped with a diffeology is called a *diffeological space*. The elements of the diffeology $\mathcal{D}$ of a diffeological space X are called the *plots* of (or in) the space.[1]

20. Note. Formally, a diffeological space is a pair $(X, \mathcal{D})$ where X is the underlying set and $\mathcal{D}$ the chosen diffeology, but we generally use a single letter to make the reading lighter. For example, we can use the letter $\mathcal{X}$ for the pair $(X, \mathcal{D})$, or anything else suggestive.

[1]There is a discussion about diffeology as a sheaf theory in [Igl87, Annex]. But we do not develop this formal point of view in general, because the purpose of diffeology is to minimize the technical tools in favour of a direct, more geometrical, intuition.

21. Example: smooth diffeology on $\mathbf{R}^n$. The first and foremost examples of diffeological spaces are the Euclidean domains. The plots of a domain $\mathcal{O}$ are all the smooth parametrizations $F: U \to \mathcal{O}$, where U is any other domain, of any dimension. We call this diffeology the *smooth diffeology*, or the *standard diffeology*. It is clear that the three axioms are satisfied. They have been chosen exactly because they are the fundamental properties which we want to replicate on sets, to define a *smooth structure*. Note that this is not the only way to imagine smooth structure on sets. We may compare diffeology with other approaches in the future.

10. Category {Diffeology}

22. Smooth maps. After defining the structure of diffeological space, the main constituent in diffeology is the notion of *smooth map*. Let X and X′ be two diffeological spaces. A map $f: X \to X'$ is said to be smooth if (and only if) the composite with any plot in X is a plot in X′, which can be summarized by

$$f \circ \mathcal{D} \subset \mathcal{D}',$$

where $\mathcal{D}$ and $\mathcal{D}'$ denotes the respective diffeologies. The set of smooth maps from X to X′ is denoted by[2]

$$\mathcal{C}^\infty(X, X') = \{f \in \operatorname{Maps}(X, X') \mid f \circ P \in \mathcal{D}',\ \forall P \in \mathcal{D}\}.$$

23. Example: smooth parametrizations. The plots of a diffeology are the first examples of smooth maps. Indeed, let $P: U \to X$ be a plot of X, let $F: V \to U$ be a plot of the smooth diffeology on the domain U, that is $F \in \mathcal{C}^\infty(V, U)$. Then, the composite $P \circ F$ is a plot of X, that is the third axiom of diffeology, the "smooth compatibiity". Therefore,

$$\mathcal{C}^\infty(U, X) = \{P \in \mathcal{D} \mid \operatorname{dom}(P) = U\}.$$

In particular, for the domains U and V, the notation $\mathcal{C}^\infty(U, V)$ is understood in the usual sense and in the diffeological sense coincide.

[2] $\operatorname{Maps}(X, X')$ denotes the set of all maps from X to X′.

Hence, there is no need to introduce a special notation to denote the plots of a diffeological space.

24. Category {Diffeology}. Consider X, X′ and X″ be three diffeological spaces, with diffeologies $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$. Let $f \in \mathcal{C}^\infty(X, X')$ and $f' \in \mathcal{C}^\infty(X', X'')$, then $f' \circ f \in \mathcal{C}^\infty(X, X'')$. Thus, diffeological spaces, together with smooth maps, define a category we write {Diffeology}.

The isomorphisms of this category are called *diffeomorphisms*, they are bijective maps, smooth as well as their inverse. In the case of a diffeomorphism $f \circ \mathcal{D} = \mathcal{D}'$. The set of diffeomorphisms from X to X′ is denoted by $\mathrm{Diff}(X, X')$.

25. Remark. {Euclidean Domains} is a full subcategory of {Diffeology}, which is a strict extension of it on sets. We could call such extensions "smooth categories", but that is just to identify the general context of the theory.

Pick, for example, the smooth $\mathbf{R}^2$: we have a diffeology on $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ by lifting locally the parameterizations in $\mathbf{R}^2$. That is, a plot of T^2 will be a parameterization $P\colon r \mapsto (z_r, z'_r)$ such that, for every point in the domain of P, there exist two smooth parameterizations θ and θ', in $\mathbf{R}$, defined in the neighbourhood of this point, with $(z_r, z'_r) = (e^{2i\pi\theta(r)}, e^{2i\pi\theta'(r)})$, that is, the usual diffeology that makes T^2 the manifold we know.

That procedure can be extended naturally to the quotient $T_\alpha = \mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z})$, where α is some number. Indeed, a parameterization $P\colon r \mapsto \tau_r$ in T_α is a plot if there exists locally, in the neighbourhood of every point in the domain of P, a parameterization $r \mapsto t_r$, such that $\tau_r = \mathrm{class}(t_r)$, with $\mathrm{class} : \mathbf{R} \to T_\alpha$ the projection.

This is exactly the diffeology we are considering when we talk about the irrational torus. This construction of diffeologies by *pushforward* is actually one of the fundamental constructions of the theory, and for that, we need to introduce an important property of the set of diffeologies of a set.

11. Order in diffeology

26. Comparing diffeologies. Inclusion defines a partial order in diffeology, called *fineness*. If $\mathcal{D}$ and $\mathcal{D}'$ are two diffeologies on a set X, one says that $\mathcal{D}$ is *finer* than $\mathcal{D}'$ if $\mathcal{D} \subset \mathcal{D}'$. We denote by

$$\mathcal{D} \preceq \mathcal{D}'.$$

We say also that $\mathcal{D}'$ is *coarser* than $\mathcal{D}$. Every set has two extreme

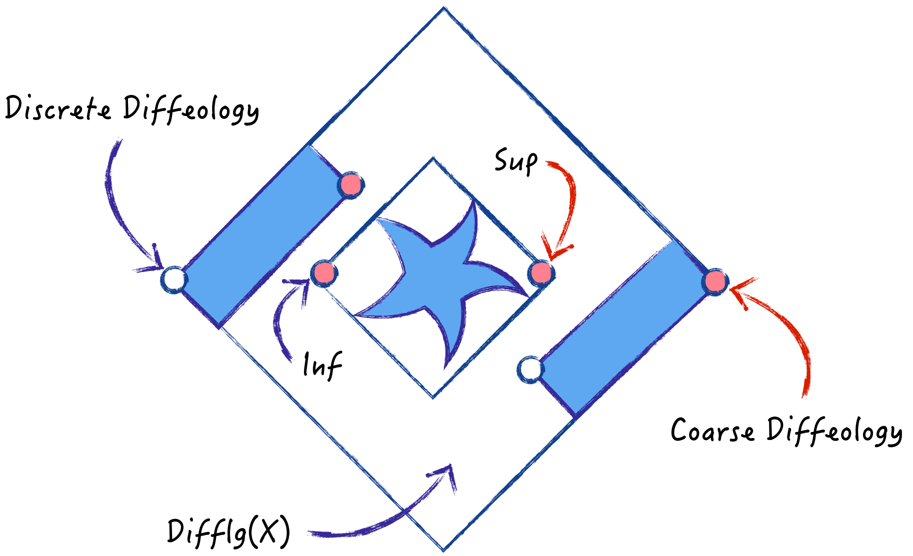


Figure 2. Comparing Diffeologies.

diffeology:

(1) The *discrete diffeology* where plots are only local constant parametrizations is the finest.
(2) The *coarse diffeology* where plots are all the parametrizations is the coarsest.

27. Infimum and supremum. Diffeologies are stable by intersection. Any family $(\mathcal{D}_i)_{i\in\mathcal{I}}$ of diffeologies on a set X has an *infimum*, it is the intersection:

$$\inf(\mathcal{D}_i)_{i\in\mathcal{I}} = \bigcap_{i\in\mathcal{I}} \mathcal{D}_i.$$

It is the coarsest diffeology which is contained in every element of the family $(\mathcal{D}_i)_{i\in\mathcal{I}}$.

The family $(\mathcal{D}_i)_{i\in\mathcal{I}}$ has also a *supremum*, it is the finest diffeology containing every element of the family:

$$\sup(\mathcal{D}_i)_{i\in\mathcal{I}} = \inf\{\mathcal{D}' \in \mathrm{Diffeologies(X)} \mid \forall i \in \mathcal{I}, \mathcal{D}_i \subset \mathcal{D}'\}$$

where Diffeologies(X) denotes the set of all diffeologies on the set X.

This partial order, fineness, makes the set of diffeologies on a set X, a *lattice*. That is, a set such that every subset of diffeologies has an infimum and a supremum.

As usual, if the infimum of a family belongs to the family, then it is called a *minimum*; and if the supremum belongs to the family, then it is called the *maximum*.

This property of being a lattice is very useful in defining diffeologies by means of properties. We shall see that most of diffeologies are the finest or coarsest diffeologies such that some property is satisfied. Because mimina and maxima are always distinguished elements in a set when they exist. The following examples will illustrate the point.

12. Pushing and pulling diffeology

28. Pushing forward diffeologies. Let $f: \mathrm{X} \to \mathrm{X}'$ be a map, and let X be a diffeological space, with diffeology $\mathcal{D}$. Then, there exists a finest diffeology on X' such that f is smooth. It is called the *pushforward* of the diffeology of X. We denote it by $f_*(\mathcal{D})$.

If f is surjective, its plots are the parameterizations P' in X' that can be written $\mathrm{Sup}_i\, f \circ \mathrm{P}_i$, where the P_i are plots of X such that the $f \circ \mathrm{P}_i$ are compatible, that is, coincide on the intersection of their domains, and Sup denotes the smallest common extension of the family $\{f \circ \mathrm{P}_i\}_{i\in\mathcal{I}}$. Formally speaking, a parametrization $\mathrm{P}': \mathrm{U} \to \mathrm{X}$ belongs to $f_*(\mathcal{D})$, if (and only if) there exists a family $(\mathrm{P}_i)_{i\in\mathcal{I}}$ of plots of X, $P_i \in \mathcal{D}$, defined on an open covering $(U_i)_{i\in\mathcal{J}}$ of U, such that $f \circ P_i = P' \restriction U_i$.

$$f_*(\mathcal{D}) = \{P' \in \mathrm{Param}(X') \mid \exists P_i \in \mathcal{D}, i \in \mathcal{J}, P' = \mathrm{Sup}_i(f \circ P_i)\}.$$

It is equivalent to say that for all $r \in U$ there exists an open neighbourhood V and a plot Q in X, such that $P \restriction V = f \circ Q$.

29. Subductions. Let $\pi\colon X \to X'$ be a map between diffeological spaces. We say that π is a *subduction* if (and only if)

(1) The map π it is surjective.
(2) The pushforward of the diffeology of X coincides with the diffeology of X'.

We can check that the composite of two subductions is again a subduction, that makes the subcategory {Subductions}.

Let's talk about quotients.

30. What a quotient is and where does it live. There is sometimes an ambiguity about the construction of quotient sets that needs to be adressed once and for all. They are too often identified with some sets of representants in a way that can be regarded as arbitrary. Let us begin with a set X and an *equivalence relation* $\sim$ on X, that is, a binary relation which is reflexive, symetric and transitive. Let $x \in X$, the *equivalence class* $\mathrm{class}(x)$ of x is by definition the subset

$$\mathrm{class}(x) = \{x' \mid x' \sim x\} \subset X.$$

It lives then naturally in the *powerset*

$$\mathrm{class}(x) \in \mathfrak{P}(X) = \{A \mid A \subset X\},$$

set of all the subsets of X. The quotient set Q of X by $\sim$, denoted generally by $X/\sim$, can always be regarded as a subset of $\mathfrak{P}(X)$:

$$Q = \{\mathrm{class}(x) \mid x \in X\} \subset \mathfrak{P}(X).$$

The *canonical projection* class is then the application

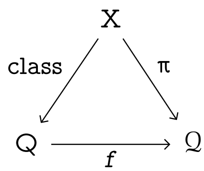
$$\mathrm{class}\colon X \to \mathfrak{P}(X) \quad \text{with} \quad X/\sim\ = \mathrm{class}(X).$$

When it comes to quotient space in these lectures, it is always the way we look at it.

31. Quotient spaces. Let X be a diffeological space and let $\sim$ be an equivalence relation on X. Let $Q = X/\sim$ be the quotient set. The pushforward $\text{class}_*(\mathcal{D})$ on Q of the diffeology of X, by the canonical projection, is called the *quotient diffeology*. Equipped with the quotient diffeology, Q is called the *quotient space* of X by $\sim$.

This is the first important property of the category {Diffeology}, it is closed by quotient, and we shall see not trivially closed.

Note that considering the quotient $Q = X/\sim$ for what it is, as described above, does not prevent us to identifiy it with some smooth representant $\mathcal{Q}$, according to the diagram where class and π are two subductions and f a bijection, and therefore a diffeomorphism.



32. Pulling back diffeologies. Let $f\colon X \to X'$ be a map, and let X' be a diffeological space with diffeology $\mathcal{D}'$. Then, there exists a coarsest diffeology on X such that f is smooth. It is called the *pullback* of the diffeology of X'. We denote it by $f^*(\mathcal{D}')$. Its plots are the parameterizations P in X such that $f \circ P$ is a plot of X'.

$$f^*(\mathcal{D}') = \{P \in \text{Param}(X) \mid f \circ P \in \mathcal{D}'\}.$$

33. Inductions. Let X and X' be two diffeological spaces and $f\colon X \to X'$ be a map. We say that f is an induction if (and only if):

(1) The map f is injective.
(2) The pullback $f^*(\mathcal{D}')$ of the diffeology of X' coincide with the diffeology $\mathcal{D}$ of X.

34. Subset diffeology. Pulling back diffeologies gives to any subset $A \subset X$, where X is a diffeological space, a *subset diffeology* $j^*(\mathcal{D})$, where $j\colon A \to X$ is the inclusion and $\mathcal{D}$ is the diffeology of X. A subset equipped with the subset diffeology is called a *diffeological subspace*. Plots of the subset diffeology are simply plots of X taking their values in the subset A.

This is a second important property of the category {Diffeology}, it is closed by inclusion.

35. Discrete subspaces. We have defined *discrete diffeological spaces* as diffeological spaces equipped with the discrete diffeology. That is, the diffeology consisting in locally constant parametrization. It happens that subspaces of diffeological spaces inherit the discret diffeology.

The best example of a discrete subset in diffeology is probably $\mathbf{Q} \subset \mathbf{R}$. This corresponds perfectly to what we understand intuitivly to be discrete. But we have to be careful because discrete in diffeology does not coincide always with discrete in topology, in particular for $\mathbf{Q}$ in $\mathbf{R}$ that topologists do not consider as discrete, which is a little bit exagerate. However, it is always preferable to specify in which sense we are using the word "discrete" when in doubt, to avoid any confusion with topologists.

Proof. Consider a plot $P\colon U \to \mathbf{R}$ but with values in $\mathbf{Q}$. Let $r, r' \in U$ and $\gamma\colon t \mapsto P(tr' + (1-t)r)$, defined on a small open neighbourhood of $[0,1]$. Then, γ is a plot in $\mathbf{R}$ and therefore continuous. Let $q = \gamma(0) = P(r)$ and $q' = \gamma(1) = P(r')$, by hypothesis $q, q' \in \mathbf{Q}$. Since γ is continuous, according to the *intermediate values theorem*,[3] γ takes every values between q and q'. if $q \neq q'$, there exists always an irrational number in between, which cannot be because P takes its values only in $\mathbf{Q}$. Therefore $\gamma(0) = \gamma(1)$, that is, $P(r) = P(r')$. The plot P is then locally constant since it will be constant on every small ball around every $r \in U$. ▶

36. Example: the circle. Let S^1 be the circle, defined by

$$S^1 = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}.$$

[3] https://www.wikiwand.com/en/Intermediate_value_theorem

It is equivalent the set $U(1)$ of complex numbers of modulus 1, $z = x + iy \in U(1)$ means[4] $\bar{z}z = x^2 + y^2 = 1$. The plots of S^1, as a diffeological subspace of $\mathbf{R}^2$, are just the pairs $r \mapsto (x(r), y(r))$ of smooth real functions defined on some Euclidean domain U, such that for all r in U, $x(r)^2 + y(r)^2 = 1$. In particular, for $r = \theta \in \mathbf{R}$:

Proposition. *The projection, from* $\mathbf{R}$ *to* S^1*,*

$$\pi \colon \theta \mapsto (\cos(\theta), \sin(\theta))$$

is a subduction.

Indeed, For any θ, one of the derivative $x'(\theta) = -\sin(\theta)$ or $y'(\theta) = \cos(\theta)$ does not vanishes, since $x'(\theta)^2 + y'(\theta)^2 = 1$. Assume that $x'(\theta_0) \neq 0$, then according to the inverse function theorem,[5] there exists a small interval $\mathcal{I}$ centered at θ_0 such that $\phi = \cos \restriction \mathcal{I}$ is a diffeomorphism onto its image, an open intervall that we denote by $\mathcal{J} = \cos(\mathcal{I})$. So, let $r \mapsto (x(r), y(r))$ be a plot in S^1 and assume that $x(r_0) = \cos(\theta_0)$ and $\sin(\theta_0) \neq 0$. The preimage $\mathcal{O} = x^{-1}(\mathcal{J})$ is an open subset in U, since $r \mapsto x(r)$ is smooth.

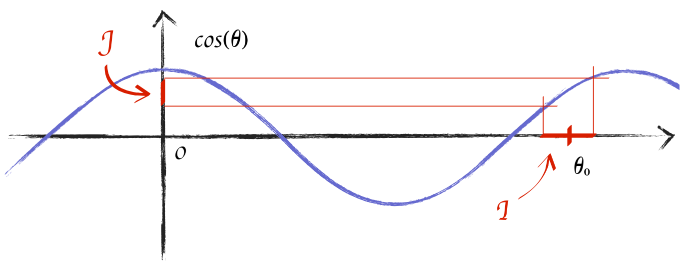


Figure 3. Function Cosinus.

Then, let $\theta(r) = \phi^{-1}(x(r))$, defined on $\mathcal{O}$, and for all $r \in \mathcal{O}$, $x(r) = \cos(\theta(r))$ and $y(r) = \sin(\theta(r))$. The map $r \mapsto \theta(r)$ is a *local lifting*, along the projection π, of the plot $r \mapsto (x(r), y(r))$.

---

[4] $\bar{z}$ or z^* denote the conjugate $x - iy$ of $z = x + iy$.

[5] https://www.wikiwand.com/en/Inverse_function_theorem

There exists then a bijection $f\colon \mathbf{R}/2\pi\mathbf{Z} \to S^1$

$$f(\mathrm{class}(x))) = (\cos(x), \sin(x)),$$

and since $\pi\colon x \mapsto (\cos(x), \sin(x))$ is a subduction, f is a diffeomorphisme satisfying $f \circ \mathrm{class} = \pi$, and therefore S^1 is a smooth representant of $\mathbf{R}/2\pi\mathbf{Z}$.

On the other hand, we can define also

$$\sigma : x \mapsto \frac{x}{2\pi} - \left[\frac{x}{2\pi}\right],$$

where the bracket denotes the integer part. Then $\sigma(x) = \sigma(x')$ if and only if $x' = x + 2\pi k$, with $k \in \mathbf{Z}$. Set theoretically, the interval $[0, 2\pi[= \sigma(\mathbf{R}) \subset \mathbf{R}$ represent $\mathbf{R}/2\pi\mathbf{Z}$, but equipped with the subset diffeology, σ is discontinuous and it cannot be a smooth representation of the quotient $\mathbf{R}/2\pi\mathbf{Z}$. Of course we can push forward the

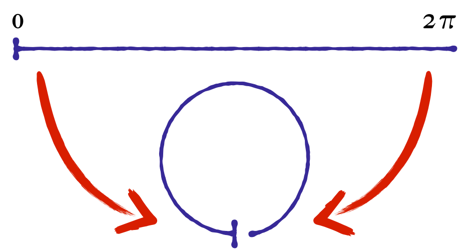


Figure 4. Closing the circle by push forward.

smooth diffeology of $\mathbf{R}$ onto the segment $[0, 2\pi[$, but that would consist to glue the end of the interval near 2π to the origin 0, and so to reconstruct the circle as shown in Figure 4.

37. Strict maps. The last example of the projection $\pi\colon \theta \mapsto (\cos(\theta), \sin(\theta))$, from $\mathbf{R}$ to $S^1 \subset \mathbf{R}^2$ suggest the definition of a new kind of map, the *strict maps*.

Every map $f\colon X \to Y$ defines the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle class}\downarrow & & \uparrow{\scriptstyle j} \\ X/f & \underset{\phi}{\longrightarrow} & f(X) \end{array}$$

where

(1) X/f denotes the quotient by the relation $f(x) = f(x')$.
(2) The map $j \colon f(X) \to X'$ is the inclusion.
(3) The map $\phi \colon X/f \to f(X)$ is defined by $\phi(\text{class}(x)) = f(x)$.

Let now X and X' be two diffeological spaces:

We say that f is *strict* if ϕ is a diffeomorphism when X/f is equipped with the quotient diffeology and $f(X)$ with the subset diffeology.

In particular, the map π above is strict. Strict maps realize quotient as subset of other diffeological spaces.

13. Making sum of diffeologies

38. Direct sum diffeology. Consider a family $(X_i)_{i \in \mathcal{I}}$ of diffeological spaces, for any family of indices. The *direct sum*, or simply the *sum* of the (elements of) family is defined by

$$\coprod_{i \in \mathcal{I}} X_i = \{(i, x) \mid i \in \mathcal{I} \text{ and } x \in X_i\}.$$

Proposition. *There exists on the sum* $X = \coprod_{i \in \mathcal{I}} X_i$ *a finest diffeology such that every injections*

$$j_i \colon X_i \to X \text{ defined by } j_i(x) = (i, x)$$

is smooth.

The plots of this diffeology are the parametrizations $r \mapsto (i(r), x(r))$ such that $r \mapsto i(r)$ is locally constant. In other words, a plot is locally with values in only one component of the sum.

Actually, the injections j_i are inductions. The space X is called the diffeological sum of the X_i.

Now, the category {Diffeology} is also closed by sum.

39. Examples: some diffeological sums. Consider

$$\mathcal{E} = \coprod_{x \in \mathbf{R}} \mathbf{R}.$$

Every element of $\mathcal{E}$ is a pair $(x, y) \in \mathbf{R}^2$, but of course the diffeology of the sum is not the smooth diffeology of $\mathbf{R}^2$. Indeed, a plot in E write always locally $r \mapsto (x, y(r))$, where x is constant and $r \mapsto y(r)$ is smooth. We could call this diffeology the "comb diffeology" of $\mathbf{R}^2$.

Another example, bigger: let $n \in \mathbf{N}$, let $x \in \mathbf{R}^n$ and $\varepsilon \in]0, \infty[$, let $\mathcal{B}(x, \varepsilon)$ be the open ball in $\mathbf{R}^n$ centered in x with radius ε. The *world of balls* would be the sum

$$\mathcal{X} = \coprod_{n \in \mathbf{N}} \coprod_{\substack{x \in \mathbf{R}^n \\ \varepsilon \in]0,\infty[}} \mathcal{B}(x, \varepsilon).$$

In diffeology, don't be afraid to think big. One can also access an

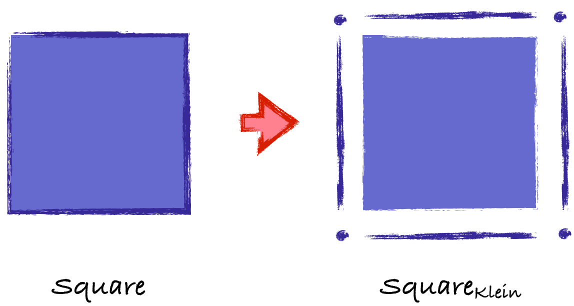


Figure 5. Klein's exploded view of the square.

aspect of the structure of a diffeological space by means of sum of parts: consider the group of diffeomorphisms Diff(X) a diffeological space X. It decomposes the space into a set of orbits $\mathcal{O} \in \mathrm{X}/\mathrm{Diff}(\mathrm{X})$. Then we reconstruct a finer diffeological space by considering the sum of the orbits, an exploded view of the space reavealing its singular structure. We can call it the Klein's exploded view.

$$\mathrm{X}_{\text{Klein}} = \coprod_{\mathcal{O} \in \mathrm{X}/\mathrm{Diff}(\mathrm{X})} \mathcal{O}.$$

14. Grow and multiply

40. Product diffeology. Consider a family $(X_i)_{i\in\mathcal{I}}$ of diffeological spaces, for any family of indices. Let pr_1 be the first projection of the direct sum of the X_i, that is

$$\mathrm{pr}_1 \colon \coprod_{i\in\mathcal{I}} X_i \to \mathcal{I} \quad \text{with} \quad \mathrm{pr}_1(i,x) = i.$$

The product of the X_i is defined as the set of section of pr_1, that is

$$\prod_{i\in\mathcal{I}} X_i = \left\{ x \colon \mathcal{I} \to \coprod_{i\in\mathcal{I}} X_i \mid \mathrm{pr}_1 \circ x = 1_{\mathcal{I}} \right\}$$

Let $X = \prod_{i\in\mathcal{I}} X_i$. An element $x \in X$ can be denoted as $x = (x_i)_{i\in\mathcal{I}}$, where $x_i = x(i)$. The projection π_i on the i-th factor X_i is defined by

$$\pi_i(x) = x_i.$$

Proposition. There exists on the product X a coarsest diffeology such that each projection π_i is smooth. Equiped with this diffeology, X is called the *diffeological product*, or simply the *product*, of the X_i.

Actually, the projections are subductions. A parametrization of the product writes $r \mapsto (x_i(r))_{i\in\mathcal{I}}$, where the x_i are plots of the X_i.

Now, the category {Diffeology} is also closed by product.

41. Examples: some diffeological products. The main example here is the power $\mathbf{R}^n$ which is the product of n copies of $\mathbf{R}$ equiped with the smooth diffeology.

A special and remarkable feature of diffeology is that the set of the smooth maps between diffeological spaces carries a natural diffeology:

42. Functional diffeology. Let X and X′ be two diffeological spaces. There exists on $\mathcal{C}^\infty(X, X')$ a coarsest diffeology such that the *evaluation map*

$$\mathrm{ev} \colon \mathcal{C}^\infty(X, X') \times X \to X' \quad \text{defined by} \quad \mathrm{ev}(f, x) = f(x),$$

is smooth. Thus diffeology is called the *functional diffeology*.

The plots of that diffeology are the parametrizations $r \mapsto f_r$, defined on some domain U, such that

$$(r, x) \mapsto f_r(x) \quad \text{from} \quad \mathrm{U} \times \mathrm{X} \quad \text{to} \quad \mathrm{X}'$$

is smooth. That means that for every plot $s \mapsto x_s$ in X, defined on some domain V, the parametrization $(r, s) \mapsto f_r(x_s)$, defined on U×V, is a plot of X′.

Now, let X, X′ and X″ be three diffeological spaces. Then,

(1) The product
$$\circ(f, g) = g \circ f,$$
defined on $\mathcal{C}^\infty(\mathrm{X}, \mathrm{X}') \times \mathcal{C}^\infty(\mathrm{X}', \mathrm{X}'')$ to $\mathcal{C}^\infty(\mathrm{X}, \mathrm{X}'')$, is smooth.

(2) The spaces $\mathcal{C}^\infty(\mathrm{X}, \mathcal{C}^\infty(\mathrm{X}', \mathrm{X}''))$ and $\mathcal{C}^\infty(\mathrm{X} \times \mathrm{X}', \mathrm{X}'')$ are diffeomorphic. The diffeomorphism ϕ consists in the game of parenthesis, for all $f \in \mathcal{C}^\infty(\mathrm{X}, \mathcal{C}^\infty(\mathrm{X}', \mathrm{X}''))$
$$\phi(f) \colon (x, x') \mapsto f(x)(x').$$

We say that the category {Diffeology} is *Cartesian closed*.

A forest, a sum of trees…

The Irrational Tori

In this lecture we will study the examples of irrational tori, quotients of torus T^n by irrational hyperplanes.

The irrational torus is the first example in diffeology that made the difference with the other generalisations of differential geometry. It appears for the first time in our paper "Exemples de groupes difféologiques: flots irrationnels sur le tore" [DI83], at the very beginning of the theory of diffeologies in 1983. It is this example that has motivated the subsequent development of the theory.

The irrational torus is a quotient space that is topologically trivial but, as it has been proven, absolutely not trivial for the quotient diffeology. We shall see in this example how its diffeology captures the maximum possible of its construction. It is also an example of how diffeology can be sensitive to arithmetic and reveal it when it is involved in a hidden way.

15. What is a torus?

The story begins with the ordinary multidimension torus T^n, which is the n-power of the 1-dimensional torus

$$T = S^1 = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\} \simeq U(1).$$

We have seen that this space, equipped with the subset diffeology of $\mathbf{R}^2$ in the previous lecture.

We recall that we have also seen that the map

$$\pi\colon \mathbf{R} \to \mathbf{R}^2 \quad \text{with} \quad \pi(t) = (cos(2\pi t), sin(2\pi t))$$

is a subduction from $\mathbf{R}$ to $S^1 \subset \mathbf{R}^2$ that identifies smoothly the quotient space $\mathbf{R}/\mathbf{Z}$ with S^1, $T \simeq \mathbf{R}/\mathbf{Z}$. The preimage of a point $z = (\cos(2\pi t), \sin(2\pi t))$ is the orbit of t by $\mathbf{Z}$, that is

$$\pi^{-1}(z) = \{t + k \mid k \in \mathbf{Z}\}.$$

The torus T is naturally a group, quotient of the additive $\mathbf{R}$ by the subgroup $\mathbf{Z}$. It is a *diffeological group* (actually, a Lie group). More-

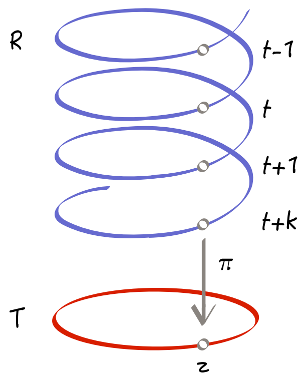


Figure 6. Covering of the Circle.

over, the projection π is the *universal covering* of T, which exists and is unique up to an isomorphism for any *connected* diffeological space. These words will be defined precisely later.

Now, the 2-torus

$$T^2 = T \times T \subset \mathbf{R}^2 \times \mathbf{R}^2$$

is the product of the torus T by itself, its square. It is equipped with the product diffeology we have seen in the previous lecture. A plot of in T^2 is a parametrization

$$r \to (z_1(r), z_2(r)) = \Big(\big(x_1(r), y_1(r)\big), \big(x_2(r), y_2(r)\big)\Big)$$

such that the x_i and y_i are smooth parametrizations such that $x_i(r)^2 + y_i(r)^2 = 1$ for all r.

Next, we can consider the square of the projection π, let us denote it just by π_2

$$\pi_2 \colon \mathbf{R}^2 \to \mathrm{T}^2$$

with

$$\pi_2(t_1, t_2) = \Big(\big(\cos(t_1), \sin(t_1)\big), \big(\cos(t_2), \sin(t_2)\big)\Big)$$

Since the projection π on each factor is a subduction from $\mathbf{R}$ onto its image $\mathrm{T} \subset \mathbf{R}^2$, the product π_2 is a subduction from $\mathbf{R}^2$ onto its image $\mathrm{T}^2 \subset (\mathbf{R}^2)^2$. Therefore the square T^2 of T identifies with the quotient

$$\mathrm{T}^2 \simeq (\mathrm{R}/\mathrm{Z})^2 = \mathrm{R}^2/\mathrm{Z}^2,$$

where $\mathbf{Z}^2 \subset \mathbf{R}^2$ is the subset of points with integer coordinates.

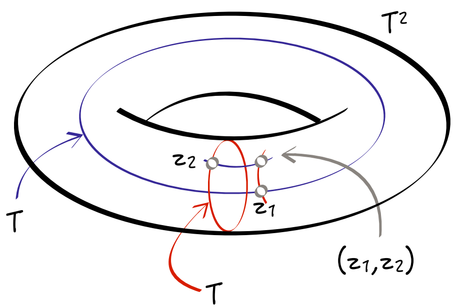


Figure 7. The 2-torus.

More generally, a n-dimensional torus T^n is the n-th power of the 1-dimensional torus T

$$\mathrm{T}^n = \{(z_1, \ldots, z_n) \mid \forall i, z_i \in \mathrm{T}\}.$$

And also equivalent to the quotient

$$\mathrm{T}^n \simeq (\mathrm{R}/\mathrm{Z})^n = \mathrm{R}^n/\mathrm{Z}^n.$$

where $\mathbf{Z}^n \subset \mathbf{R}^n$ is the subgroup of points with integer coordinates. Again, T^n is a diffeological group (a Lie group more precisely), an Abelian one.

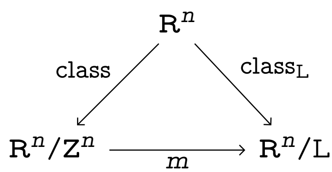
Remark. Consider a *lattice* in $\mathbf{R}^n$, that is, a subgroup like

$$L = \left\{ \sum_{i=1}^{n} n_i v_i \mid n_i \in \mathbf{Z} \right\},$$

where the $(v_i)_{i=1}^n$ are a basis of $\mathbf{R}^n$. Then the quotient space $\mathbf{R}^n/L$ is naturally difféomorphic to T^n. Indeed, let $M\colon \mathbf{R}^n \to \mathbf{R}^n$ be the linear isomorphism $M(x) = \sum_{i=1}^n x_i v_i$, with $x = (x_1, \ldots, x_n)$. The map

$$m = \mathrm{class}(x) \mapsto \mathrm{class}_L(x)$$

is well defined and defines a smooth group isomorphism from $T^n = \mathbf{R}^n/\mathbf{Z}^n$ to $\mathbf{R}^n/L$.



So, diffeologically speaking there is only one torus T^n: all lattices are equivalent.

The various tori are often described as the power of the unitary group

$$U(1) = \{z \in \mathbf{C} \mid \bar{z}z = 1\},$$

where $\bar{z}$ denotes z conjugate. Thus,

$$T^n \simeq U(1)^n = \{(z_1, \ldots, z_n) \mid \forall i, z_i \in U(1)\}$$

There, the group law is just the pointwise multiplication:

$$(z_1, \ldots, z_n) \cdot (z'_1, \ldots, z'_n) = (z_1 z'_1, \ldots, z_n z'_n).$$

We remark that the multiplication is smooth, that means that for two plots $r \mapsto (z_1(r), \ldots, z_n(r))$ and $r \mapsto (z'_1(r), \ldots, z'_n(r))$, defined on the same domain, the resulting parametrization $r \mapsto (z_1(r)z'_1(r), \ldots, z_n(r)z'_n(r))$ is again a plot in T^n. The inversion $r \mapsto (\bar{z}_1(r), \ldots, \bar{z}_n(r))$ also is smooth. We say that T^n is a *diffeological group*. We shall develop later a little bit about diffeological group, especially when it will come to the moment map and symplectic diffeology. But for now, that is all we need.

16. The irrational torus $\mathbb{T}_\alpha$

The object "irrational torus" has been motivated by physics, by a question related to the behavior of a particle submited to a quasiperiodic potential. These quasiperiodic potentials describe the phenomenon of a quasiperiodic pattern in cristals. For example, the Figure 8 is representing the diffraction of an aluminium-palladium-manganese (Al-Pd-Mn) quasicrystal surface.

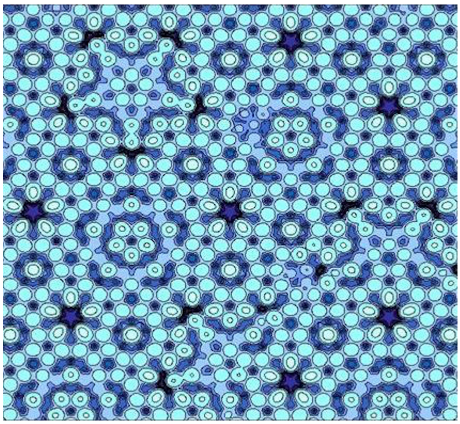


Figure 8. A Diffraction Figure of a Quasicristal.

For this type of material, the diffraction pattern is not periodic as it is usually for a crystal, i.e. it does not draw a periodic tiling of the plane, but something close without being quite so.

The physicists and the mathematicians who were involved in these researchs decided that, that phenonenom could be described by a *quasiperiodic potential*. I will try to outline their approach without being able to be too precise.

In classical physics, the motion of a particle in a medium is described by a force which is the gradient of a real function called the potential.

So, let us consider the simplest example, a toy model: a particle moving on a line submited to a force that is the derivative of a real function $V : \mathbf{R} \to \mathbf{R}$, which is assumed to be smooth. Physicists are interested in the spectrum of the so-called (quantum) Hamiltonian:

$$\hat{H} = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

which is an operator on some Hilbert space of functions. Two main special cases are illustrated by figure 9.

(1) The *periodic case* is described by the potential

$$V_1 : x \mapsto U_1\left(e^{2i\pi x}\right)$$

where U_1 is defined on the circle S^1.

(2) The *quasiperiodic case* is described by the potential

$$V_2 : x \mapsto U_2 \circ j_\alpha(x),$$

where U_2 is a function defined on the 2-torus and $j_\alpha : \mathbf{R} \to T^2$ is the map

$$j_\alpha : x \mapsto \left(e^{2i\pi x}, e^{2i\pi\alpha x}\right) \quad \text{with} \quad \alpha \in \mathbf{R} - \mathbf{Q}.$$

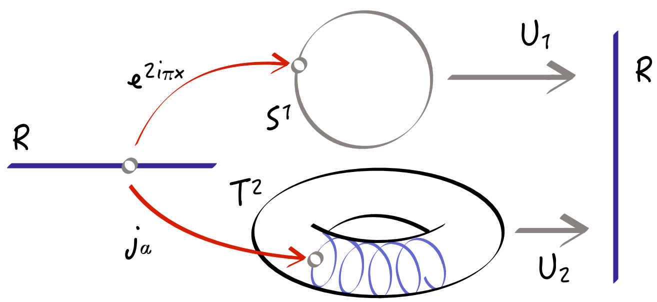


Figure 9. Periodic and Quasiperiodic Potential.

So, the quasiperiodic property is encoded in the *irrational solenoid*

$$\mathcal{S}_\alpha = \left\{\left(e^{2i\pi x}, e^{2i\pi\alpha x}\right) \mid x \in \mathbf{R}\right\}.$$

We remark first that $\mathcal{S} \subset T^2$ is a subgroup.

Our intention now is not to solve the general question of the spectrum of the Hamiltonian in presence of quasiperiodic potential, but to delve deeper into issues surrounding these context. In particular:

43. Definition. *We call* irrational torus T_α *the quotient space*

$$T_\alpha = T^2/\mathcal{S}_\alpha,$$

equipped with the quotient diffeology.

44. Proposition. The map $j_\alpha \colon x \mapsto \left(e^{2i\pi x}, e^{2i\pi\alpha x}\right)$ is an induction from $\mathbf{R}$ into T^2, with image the solenoid $\mathcal{S}_\alpha$.

Note. We shall see further on, $\mathcal{S}_\alpha \subset T^2$ is a submanifold in the sense of diffeological manifolds, but not exactly in the usual sense because it is not embedded. In ordinary differential geometry textbooks, submanifolds are defined only embedded.

☞ Proof. Let us begin to check that the map $\pi^2 : (x, y) \mapsto (\pi(x), \pi(y))$, where $\pi(t) = (\cos(2\pi t), \sin(2\pi t))$, from $\mathbf{R}\times\mathbf{R}$ to $\mathbf{R}^2\times\mathbf{R}^2$ is strict. First of all, the map π^2 is smooth. Then, according to the definition, π^2 is strict if and only if

$$\mathrm{class}(x, y) \mapsto \Big(\big(\cos(2\pi x), \sin(2\pi x)\big), \big(\cos(2\pi y), \sin(2\pi y)\big)\Big)$$

is an induction, from $\mathbf{R}^2/\mathbf{Z}^2$ to $\mathbf{R}^2 \times \mathbf{R}^2$, with class : $\mathbf{R}^2 \to \mathbf{R}^2/\mathbf{Z}^2$. We have already seen that $\pi : t \mapsto (\cos(2\pi t), \sin(2\pi t))$ is strict, and π^2 is just the square of π. Thus, a plot $\Phi : U \to S^1 \times S^1 \subset \mathbf{R}^2 \times \mathbf{R}^2$ is just a pair of plots P and Q from U to S^1, which can be individually smoothly lifted locally along π, and give a local lift of π^2 itself. Thus, π^2 is strict.

Now, let Δ_α be the line in $\mathbf{R}\times\mathbf{R}$ with splope α, the subset of points $(x, \alpha x) \in \mathbf{R}^2$. Since α is irrational, $\pi^2_\alpha = \pi^2 \restriction \Delta_\alpha$ is injective. Indeed, $\pi^2(t, \alpha t) = \pi^2(t', \alpha t')$ means, on the one hand, $(\cos(2\pi t'), \sin(2\pi t')) = (\cos(2\pi t), \sin(2\pi t))$, and on the other hand, $(\cos(2\pi\alpha t'), \sin(2\pi\alpha t')) = (\cos(2\pi\alpha t), \sin(2\pi\alpha t))$. That is, $t' = t + k$ and $\alpha t' = \alpha t + k'$ with $k, k' \in \mathbf{Z}$, which gives $\alpha k - k' = 0$, but $\alpha \neq \mathbf{Q}$, thus $k = k' = 0$ and then $t' = t$.

Let $\Phi : U \to \mathcal{S}_\alpha \subset S^1\times S^1 \subset \mathbf{R}^2\times\mathbf{R}^2$ be a plot, with $\Phi(r) = (P(r), Q(r))$. Since π^2 is strict, for all $r \in U$, there exists locally a smooth lift $r' \mapsto (x(r'), y(r'))$ in $\mathbf{R}^2$, defined on a neighborhood V of r, such that $\pi^2(x(r'), y(r')) = (P(r'), Q(r'))$ Thus, $\pi^2(x(r'), y(r')) \in \mathcal{S}_\alpha$ for all $r' \in$

V. But, $r' \mapsto (x(r'), \alpha x(r')) \in \Delta_\alpha \subset \mathbf{R}^2$ is smooth, and $\pi^2(x(r'), \alpha x(r'))$ belongs to $\mathcal{S}_\alpha$ too. Therefore, there exists $r' \mapsto k(r') \in \mathbf{Z}$ such that $y(r') = \alpha x(r') + k(r')$, that is, $k(r') = y(r') - x(r')$. Thus, $r' \mapsto k(r')$ is smooth and takes its values in $\mathbf{Z}$, hence $k(r') = k$ constant. Then, $r' \mapsto (x(r'), y(r') - k)$ is a plot of $\mathcal{S}_\alpha$ with $\pi^2(x(r'), y(r') - k) = (\mathrm{P}(r'), \mathrm{Q}(r'))$, thus $\pi^2_\alpha : \Delta_\alpha \to \mathcal{S}_\alpha$ is an injective subduction, that is, a diffeomorphism from Δ_α to $\mathcal{S}_\alpha$, and therefore an induction. ▶

45. Proposition. The quotient space $\mathrm{T}_\alpha = \mathrm{T}^2/\mathcal{S}_\alpha$ is diffeomorphic to the quotient $\mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z})$, and isomorphic as a group.

Note 1. It is clear now that T_α, as a quotient topological space, is trivial since $\mathbf{Z} + \alpha\mathbf{Z} \subset \mathbf{R}$ is dense.

Note 2. T_α is also isomorphic to the intermediate quotient $\mathbf{R}^2/\mathbf{Z}^2(\Delta_\alpha)$, where $\mathbf{Z}^2(\Delta_\alpha)$ is the image of the line Δ_α by $\mathbf{Z}^2$, that is, the set of points $(x + n, \alpha x + m)$ with $x \in \mathbf{R}$ and $(n, m) \in \mathbf{Z}^2$.

Proof. We begin to prove that with $\alpha \neq \mathbf{Q}$, $\mathbf{Z} + \alpha\mathbf{Z}$ is dense in $\mathbf{R}$. We remark first that $\mathbf{Z} + \alpha\mathbf{Z}$ is a subgroup of $(\mathbf{R}, +)$. Let $\Gamma \subset \mathbf{R}$ be a subgroup not reduced to $\{0\}$. It is relatively obvious that: either there exists a smallest element $a \in \Gamma$ and $\Gamma = a\mathbf{Z}$, or Γ is dense. Now, if $\mathbf{Z} + \alpha\mathbf{Z} = a\mathbf{Z}$, then $\alpha = ka$ and $1 = \ell a$ with $k, \ell \in \mathbf{Z}$, that would mean that $\alpha = k/\ell$ which is not the case. Thus, $\mathbf{Z} + \alpha\mathbf{Z}$ is dense.

Let $\phi \colon \mathbf{R}^2/\mathbf{Z}^2 \to \mathrm{S}^1 \times \mathrm{S}^1$ be the identification given by the factorization of the strict map $\pi^2 \colon \mathbf{R}^2 \to \mathrm{S}^1 \times \mathrm{S}^1$. Then, the quotient $(\mathrm{S}^1 \times \mathrm{S}^1)/\mathcal{S}_\alpha = \phi(\mathbf{R}^2/\mathbf{Z}^2)/\mathcal{S}_\alpha$, is equivalent to $\mathbf{R}^2/[\mathbf{Z}^2(\Delta_\alpha)]$ where the equivalence relation is defined by the action of the subgroup $\mathbf{Z}^2(\Delta_\alpha)$. Let $\rho : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $\rho(x, y) = (0, y - \alpha x)$, it is obviously a projector, $\rho \circ \rho = \rho$, and clearly $\mathrm{class} \circ \rho = \mathrm{class}$, with $\mathrm{class} : \mathbf{R}^2 \to \mathbf{R}^2/[\mathbf{Z}^2(\Delta_\alpha)]$. Now, let $\mathrm{X}' = \mathrm{val}(\rho)$, that is, $\mathrm{X}' = \{0\} \times \mathbf{R}$. The restriction to X' of the equivalence relation defined by the action of $\mathbf{Z}^2(\Delta_\alpha)$ on $\mathbf{R}^2$, is given by the following action of $\mathbf{Z}^2$, $(n, m) \colon (0, y) \mapsto (0, y + m - \alpha n)$. Therefore, the quotient $(\mathrm{S}^1 \times \mathrm{S}^1)/\mathcal{S}_\alpha$ is equivalent to $\mathrm{X}'/(\mathbf{Z} + \alpha\mathbf{Z})$, that is, equivalent to $\mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z}) = \mathrm{T}_\alpha$. ▶

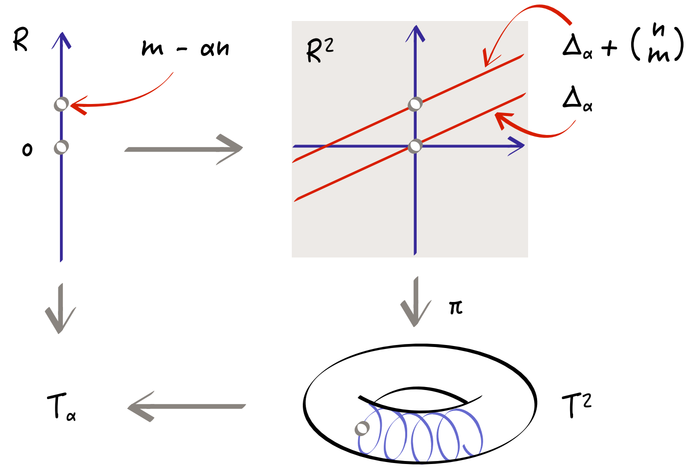


Figure 10. T_α as quotients.

46. Smooth maps from T_α to T_β. For any pair α and β of irrational numbers, the set $\mathcal{C}^\infty(T_\alpha, T_\beta)$ does not reduce to the constant maps if and only if there exists $a, b, c, d \in \mathbf{Z}$ such that

$$\alpha = \frac{c + d\beta}{a + b\beta}.$$

Note that, since α and β are irrational, the relation above has an inverse $\beta = (a\alpha - c)/(d - b\alpha)$.

Proof. Let $f : T_\alpha \to T_\beta$ be a smooth map. Consider the commutative diagram

$$\begin{array}{ccc} \mathbf{R} & \overset{F}{\dashrightarrow} & \mathbf{R} \\ {\scriptstyle \mathrm{class}_\alpha}\downarrow & & \downarrow{\scriptstyle \mathrm{class}_\beta} \\ T_\alpha & \underset{f}{\longrightarrow} & T_\beta \end{array}$$

Since class_α is a plot in T_α, $f \circ \mathrm{class}_\alpha$ is a plot of T_β. Hence, for every real x_0 there exists an open interval V centered at x_0, and a smooth parametrization $F : V \to \mathbf{R}$ such that $\mathrm{class}_\beta \circ F = (f \circ \mathrm{class}_\alpha) \restriction V$. For all real numbers x and all pairs (n, m) of integers such that

$x + n + \alpha m \in V$, there exist two integers n' and m' such that

$$F(x + n + \alpha m) = F(x) + n' + \beta m'. \qquad (\spadesuit)$$

Since β is irrational, for every such x, n and m, the pair (n', m') is unique.

Now, there exists an interval $\mathcal{J} \subset V$ centered at x_0 and an interval $\mathcal{O}$ centered at 0 such that: for every $x \in \mathcal{J}$ and for every $n + \alpha m \in \mathcal{O}$, $x + n + \alpha m \in V$. Since F is continuous and since $\mathbf{Z} + \alpha\mathbf{Z}$ is diffeologically

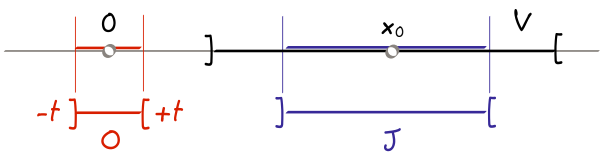


Figure 11. Intervals $V, \mathcal{O}, J$.

discrete, $n' + \beta m' = F(x + n + \alpha m) - F(x)$ is constant as function of x. But F is smooth, the derivative of the identity $(\spadesuit)$, with respect to x, at the point x_0, gives $F'(x_0 + n + \alpha m) = F'(x_0)$. Then, since α is irrational, $\mathbf{Z} + \alpha\mathbf{Z} \cap \mathcal{O}$ is dense in $\mathcal{O}$, and since F' is continuous, $F'(x) = F'(x_0)$, for all $x \in \mathcal{J}$. Hence, F restricted to $\mathcal{J}$ is affine, there exist two numbers λ and μ such that

$$F(x) = \lambda x + \mu \quad \text{for all} \quad x \in \mathcal{J}. \qquad (\clubsuit)$$

Note that, by density of $\mathbf{Z} + \alpha\mathbf{Z}$, $\text{class}_\alpha(\mathcal{J}) = T_\alpha$. Hence F defines completely the function f.

Now, applying $(\spadesuit)$ to the expression $(\clubsuit)$ of F, we get for all $n + \alpha m \in \mathcal{O}$: $\lambda(x + n + \alpha m) + \mu = \lambda x + \mu + n' + \beta m'$, that is:

$$\lambda \times (n + \alpha m) \in \mathbf{Z} + \beta\mathbf{Z}, \quad \text{that is:} \quad \lambda(\mathbf{Z} + \alpha\mathbf{Z}) \subset \mathbf{Z} + \beta\mathbf{Z}. \qquad (\blacklozenge)$$

Let us show that actually $(\blacklozenge)$ is satisfied for all $n + \alpha m$ in $\mathbf{Z} + \alpha\mathbf{Z}$. Let $\mathcal{O} =]-t, t[$, and let us take t not in $\mathbf{Z} + \alpha\mathbf{Z}$, even if we have to shorten $\mathcal{O}$ a little. Let $x \in \mathbf{Z} + \alpha\mathbf{Z}$, and $x > t$. There exists $N \in \mathbf{N}$ such that

$$0 < (N-1)t < x < Nt, \quad \text{and then} \quad 0 < \frac{x}{N} < t.$$

Now, by density of $\mathbf{Z} + \alpha\mathbf{Z}$ in $\mathbf{R}$,

$$\forall \eta > 0, \quad \exists y > 0 \quad \text{such that} \quad y \in \mathbf{Z} + \alpha\mathbf{Z} \quad \text{and} \quad 0 < \frac{x}{N} - y < \eta.$$

Choosing $\eta < t/N$ we have

$$\eta < \frac{t}{N} \quad \Rightarrow \quad 0 < x - Ny < N\eta < t \quad \text{and} \quad 0 < y < \frac{x}{N} < t.$$

Hence,

$$x, y \in \mathbf{Z} + \alpha\mathbf{Z} \quad \Rightarrow \quad x - Ny \in \mathbf{Z} + \alpha\mathbf{Z},$$

and

$$x - Ny < t \quad \Rightarrow \quad x - Ny \in \mathbf{Z} + \alpha\mathbf{Z} \cap \mathcal{O}.$$

Thus,

$$\lambda \times (x - Ny) = \lambda x - N \times (\lambda y) \in \mathbf{Z} + \beta\mathbf{Z}.$$

But,

$$y \in \mathbf{Z} + \alpha\mathbf{Z} \cap \mathcal{O} \quad \Rightarrow \quad \lambda y \in \mathbf{Z} + \beta\mathbf{Z} \quad \Rightarrow \quad N \times (\lambda y) \in \mathbf{Z} + \beta\mathbf{Z},$$

therefore, $\lambda x - N \times (\lambda y) \in \mathbf{Z} + \beta\mathbf{Z}$, together with $N \times (\lambda y) \in \mathbf{Z} + \beta\mathbf{Z}$, implies

$$\forall x \in \mathbf{Z} + \alpha\mathbf{Z}, \quad \lambda x \in \mathbf{Z} + \beta\mathbf{Z}.$$

Now, applying successively (♦) to $x = 1$ and $x = \alpha$, we get

$$\lambda \in \mathbf{Z} + \beta\mathbf{Z} \quad \text{and} \quad \lambda\alpha \in \mathbf{Z} + \beta\mathbf{Z}$$

Let

$$\lambda = a + b\beta. \quad \text{and} \quad \lambda\alpha = c + d\beta.$$

If $\lambda \neq 0$, then

$$\alpha = \frac{c + d\beta}{a + b\beta}.$$

Let us remark that, since $\mathrm{class}_\alpha(\mathcal{J}) = T_\alpha$, the map F, extended to the whole $\mathbf{R}$, still satisfies $\mathrm{class}_\beta \circ F = f \circ \mathrm{class}_\alpha$. ▶

47. Diffeomorphisms between T_α and T_β. Let α and β be two irrational numbers. The tori T_α and T_β are difeomorphic if and only if there exists $a, b, c, d \in \mathbf{Z}$ such that

$$\alpha = \frac{c + d\beta}{a + b\beta} \quad \text{with} \quad ad - bc = \pm 1.$$

We say α and β are conjugated modulo $\mathrm{GL}(2, \mathbf{Z})$ [DI83].

☞ Proof. The map f is surjective is equivalent to $\lambda \neq 0$. Let us express that f is injective: let $\tau = \text{class}_\alpha(x)$ and $\tau' = \text{class}_\alpha(x')$. The map f is injective if $f(\tau) = f(\tau')$ implies $\tau = \tau'$, that is, $x' = x + n + \alpha m$, for some relative integers n and m. Using the lifting F, this is equivalent to:

> If there exist two integers n' and m' such that $F(x') = F(x) + n' + \beta m'$, then there exist two integers n and m such that $x' = x + n + \alpha m$.

But $F(x) = \lambda x + \mu$, with $\lambda \times (\mathbf{Z} + \alpha\mathbf{Z}) \subset \mathbf{Z} + \beta\mathbf{Z}$. Hence, the injectivity writes:

> If $\lambda x' + \mu = \lambda x + \mu + n' + \beta m'$, then $x' = x + n + \alpha m$.

Which is equivalent to:

> If $\lambda y \in \mathbf{Z} + \beta\mathbf{Z}$, then $y \in \mathbf{Z} + \alpha\mathbf{Z}$.

Finally equivalent to:

$$\frac{1}{\lambda} \times (\mathbf{Z} + \beta\mathbf{Z}) \subset \mathbf{Z} + \alpha\mathbf{Z}.$$

Now, let us consider the multiplication by λ, as a $\mathbf{Z}$-linear map, from the $\mathbf{Z}$-module $\mathbf{Z}+\alpha\mathbf{Z}$ to the $\mathbf{Z}$-module $\mathbf{Z}+\beta\mathbf{Z}$, defined in the respective basis $(1, \alpha)$ and $(1, \beta)$, by

$$\lambda \times 1 = a + b \times \beta \quad \text{and} \quad \lambda \times \alpha = c + d \times \beta.$$

The two modules being identified, by their basis, to $\mathbf{Z} \times \mathbf{Z}$, the multiplication by λ and the multiplication by $1/\lambda$ are represented by the matrices

$$\lambda \simeq L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \frac{1}{\lambda} \simeq L^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The matrix L is then invertible as a matrix with coefficients in $\mathbf{Z}$, that is, $ad - bc = \pm 1$ and $L = GL(2, \mathbf{Z})$. ▶

48. The space $\mathcal{C}^\infty(T_\alpha, T_\beta)$. Every matrix $M \in L(2, \mathbf{Z})$ maps the lattice $\mathbf{Z}^2$ into itself, and the line $y = \alpha x$ is mapped into a line $y = \beta x$, that is,

$$\mathrm{M}\begin{pmatrix}1\\ \alpha\end{pmatrix} \propto \begin{pmatrix}1\\ \beta\end{pmatrix}, \quad \text{that is} \quad \mathrm{M}\Delta_\alpha = \Delta_\beta.$$

Let

$$\mathrm{M} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and α and β will be related by the same relation as above:

$$\beta = \frac{a\alpha - c}{d - b\alpha}, \quad \text{that is} \quad \alpha = \frac{c + d\beta}{a + b\beta}$$

Now, since M preserves the lattice $\mathbf{Z}^2$ and maps the line Δ_α to the line Δ_β, it defines by projection a morphism Φ of T^2, mapping the solenoid $\mathcal{S}_\alpha$ to the solenoid $\mathcal{S}_\beta$. That defines a morphism f_M from the quotient $\mathrm{T}_\alpha = \mathrm{T}^2/\mathcal{S}_\alpha$ to $\mathrm{T}_\beta = \mathrm{T}^2/\mathcal{S}_\beta$. Composed with a constant map we

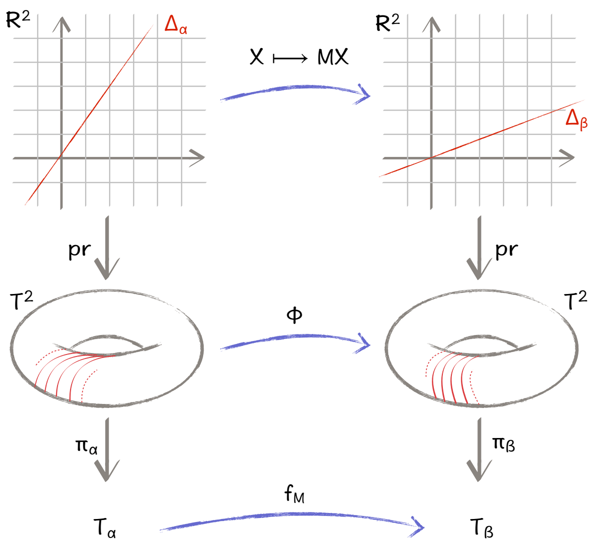


Figure 12. Linear morphism from T_α to T_β.

obtain all the smooth maps from T_α to T_β, in additive notation:

$$f : \tau \mapsto f_\mathrm{M}(\tau) + \nu.$$

In other words:

49. Proposition. Every smooth map $f\colon T_\alpha \to T_\beta$ is the projection of an affine map

$$F\colon X \to MX + N \quad \text{with} \quad M \in L(2, \mathbf{Z}) \quad \text{and} \quad N \in \mathbf{R}^2.$$

In particular, the congruence modulo $GL(2, \mathbf{Z})$ between α and β, in case of diffeomorphism, is the optimum condition we can hope for a good theory of quotients. What is remarkable here is that this is the sufficient and necessary condition in the framework of diffeology. Diffeology optimaly discriminates the irrational tori.

Now, consider the set of lines in $\mathbf{R}^2$, denoted usually by $P_2(\mathbf{R})$. Each line Δ_α defines a torus T_α.

50. Proposition. The class of equivalent irrational tori are in bijection with the orbits of the irrational lines by $GL(2, \mathbf{Z})$.

Note that this proposition extends to any quotients $T_\alpha = T^2/\mathcal{S}_\alpha$, even if $\alpha \in \mathbf{Q}$, in that case T_α is diffeomorphic to the circle S^1. The rationnal lines are one orbit under $GL(2, \mathbf{Z})$.

51. Remark: $\mathcal{C}^\infty(T_\alpha, T_\beta)$ as bimodule. Let us come back to the lifting on $\mathbf{R}$ of the smooth maps from T_α to T_β,

$$\begin{array}{ccc}
x & \xrightarrow{\quad F \quad} & \lambda x + \mu \\
{\scriptstyle \mathrm{class}_\alpha}\downarrow & & \downarrow{\scriptstyle \mathrm{class}_\beta} \\
\mathrm{class}_\alpha(x) & \xrightarrow[\quad f \quad]{} & \mathrm{class}_\beta(\lambda x) + \mathrm{class}_\beta(\mu)
\end{array}$$

Since T_β is a group, the set $\mathcal{C}^\infty(T_\alpha, T_\beta)$ is a group for the addition. The mapping

$$j\colon f \mapsto (\lambda, \rho) \quad \text{with} \quad \rho = \mathrm{class}_\beta(\mu),$$

is a group homomorphism. The map j is injective and identifies

$$\mathcal{C}^\infty(T_\alpha, T_\beta) \simeq \Lambda_{\alpha\beta} \times T_\beta,$$

with

$$\Lambda_{\alpha\beta} = \{\lambda \mid \lambda(\mathbf{Z} + \alpha\mathbf{Z}) \subset \mathbf{Z} + \beta\mathbf{Z}\}.$$

We can note here that the linear smooth maps on T_α act on the left on $\mathcal{C}^\infty(T_\alpha, T_\beta)$, and the linear smooth maps on T_β act on the right. We can denote that by

$$\Lambda_{\alpha\alpha} \cdot \Lambda_{\alpha\beta} \cdot \Lambda_{\beta\beta} \subset \Lambda_{\alpha\beta}.$$

That would correspond to

$$x \mapsto \nu x \mapsto \lambda(\nu x) + \rho \mapsto \varepsilon(\lambda(\nu x)) + \rho,$$

where $\nu(\mathbf{Z} + \alpha\mathbf{Z}) \subset \mathbf{Z} + \alpha\mathbf{Z}$ and $\varepsilon(\mathbf{Z} + \beta\mathbf{Z}) \subset \mathbf{Z} + \beta\mathbf{Z}$. These actions are commutative and make $\mathcal{C}^\infty(T_\alpha, T_\beta)$ a bimodule. <u>But this bimodule is not trivial only if α or β are quadratic numbers</u>, or both. Indeed, $\nu(\mathbf{Z} + \alpha\mathbf{Z}) \subset \mathbf{Z} + \alpha\mathbf{Z}$ implies there exists four integers $a, b, c, d \in \mathbf{Z}$ such that

$$\alpha = \frac{a + b\alpha}{c + d\alpha} \quad \Rightarrow \quad d\alpha^2 + (c - b)\alpha - a = 0.$$

Yet, still much need to be clarified here.

☞ Proof. We just prove that the map j is injective. Let $\text{class}_\beta(\lambda x) + \rho = \text{class}_\beta(\lambda' x) + \rho'$, for $x = 0$ we get $\rho = \rho'$, and then $\text{class}_\beta\big((\lambda - \lambda')x\big) = 0$ for all $x \in \mathbf{R}$. That is, $(\lambda - \lambda')x \in \mathbf{Z} + \beta\mathbf{Z}$ for all $x \in \mathbf{R}$, and thus $\lambda = \lambda'$. ▶

<u>**52. Remark: Component of $\mathcal{C}^\infty(T_\alpha, T_\beta)$.**</u> We have seen that $\mathcal{C}^\infty(T_\alpha, T_\beta)$ is isomorphic to $\Lambda_{\alpha\beta} \times T_\beta$, equiped with the functional diffeology the subgroup $\Lambda_{\alpha\beta} \times \{0\}$ is discrete, it represents the connected components, what we shall denote later by

$$\pi_0\big(\mathcal{C}^\infty(T_\alpha, T_\beta)\big) = \Lambda_{\alpha\beta}.$$

What we know better is the group of components of the group $\text{Diff}(T_\alpha)$. That is, the set of numbers $\lambda \in \mathbf{R}$ such that:

$$\lambda(\mathbf{Z} + \alpha\mathbf{Z}) \subset \mathbf{Z} + \alpha\mathbf{Z} \quad \text{and} \quad \frac{1}{\lambda}(\mathbf{Z} + \alpha\mathbf{Z}) \subset \mathbf{Z} + \alpha\mathbf{Z}$$

Considering the basis $(1, \alpha)$ of the $\mathbf{Z}$-module $\mathbf{Z}+\alpha\mathbf{Z}$, we define a, b, c, d by:

$$\lambda \times 1 = a + b\alpha \quad \text{and} \quad \lambda \times \alpha = c + d\alpha.$$

The map F lifting f associated with λ for $\rho = 0$ is represented by the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Z}),$$

and it satisfies:

$$F(x) = (a + b\alpha)x \quad \text{with} \quad \alpha = \frac{c + d\alpha}{a + b\alpha} \quad \text{and} \quad ad - bc = \pm 1. \qquad (\clubsuit)$$

As we said, except for the obvious solution $\lambda = 1$ which correspond to the inversion $x \mapsto -x$, there are no other solutions except in the case of α is quadratic.

Let us remark now that if two matrices M and M$'$ representing λ in GL(2, Z), then they are equal. Indeed,

$$\lambda = \lambda' \quad \Rightarrow \quad a + b\alpha = a' + b'\alpha \quad \Rightarrow \quad a = a' \quad \text{and} \quad b = b'.$$

Then,

$$\alpha = \frac{c + d\alpha}{a + b\alpha} = \frac{c' + d'\alpha}{a' + b'\alpha} \quad \Rightarrow \quad c + d\alpha = c' + d'\alpha$$
$$\Rightarrow \quad c = c' \quad \text{and} \quad d = d'.$$

Hence $M = M'$.

Proposition. *The set of components of* $\mathrm{Diff}(T_\alpha)$ *is isomorphic to the stabilizer, in* $\mathrm{GL}(2, \mathbf{Z})$*, of the line* Δ_α*:*

$$\pi_0(\mathrm{Diff}(T_\alpha)) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \right\}$$

According to a Dirichlet famous theorem, that we shall see in full generality in the next section, we have:

Theorem. *The group of components of* $\mathrm{Diff}(T_\alpha)$ *is isomorphic to* $\{\pm 1\} \times \mathbf{Z}$ *if* α *is quadratic, otherwise it is reduced to* $\{\pm 1\}$.

17. The general codimension 1 case

The case presented here of an irrational hyperplane in the torus T^n is the result of a joint work with Gilles Lachaud, published in 1990 [**IL90**]. The arithmetic material for this part can be found in [**BC67**].

We consider the standard torus $\mathrm{T}^n = \mathbf{R}^n/\mathbf{Z}^n$.

<u>53. Definition.</u> *We call* irrational hyperplane *in* $\mathbf{R}^n$ *an hyperplane* H *that does not contain any integer points except* 0

$$\mathrm{H} \cap \mathbf{Z}^n = \{0\}.$$

An hyperplane is directed by a linear 1*-form that is in our case normalized as follow:*

$$\mathrm{H} = \ker(w = (1\ w_2\ \ldots\ w_n)) = \{x \in \mathbf{R}^n \mid w(x) = \sum_{i=1}^{n} w_i x_i = 0\}.$$

The fact that the hyperplane is irrational is equivalent to the property of the coefficients w_i *to be* <u>*independent over* $\mathbf{Q}$</u>*:*

$$\forall q_i \in \mathbf{Q}, \quad \sum_{i=1}^{n} w_i q_i = 0 \quad \Rightarrow \quad q_i = 0, \forall i.$$

Let us denote by $\mathcal{S}_\mathrm{H} \subset \mathrm{T}^n$ *the image of* H *by the canonical projection* $\pi \colon \mathbf{R}^n \to \mathrm{T}^n$*. Here again the map* $\pi \restriction \mathrm{H}$ *is an induction.*

We define the irrational torus associated with H *as the quotient space*

$$\mathrm{T}_\mathrm{H} = \mathrm{T}^n/\mathcal{S}_\mathrm{H},$$

which is also an Abelian group.

<u>54. Proposition.</u> *The space* T_H *is diffeomorphic to the quotient:*

$$\mathrm{T}_\mathrm{H} \simeq \mathbf{R}/w(\mathbf{Z}^n).$$

where

$$w(\mathbf{Z}^n) = \{n_1 + \sum_{i=2}^{n} w_i n_i \mid n_i \in \mathbf{Z}\}$$

is a subgroup of $(\mathbf{R}, +)$*.*

<u>55. The group $\mathrm{Diff}(\mathrm{T}_\mathrm{H})$.</u> *The group of diffeomorphisms of the irrational torus* T_H *is given by*

$$\mathrm{Diff}(\mathrm{T}_\mathrm{H}) \simeq \Lambda_w \times \mathrm{T}_\mathrm{H}.$$

with Λ_w *its group of components* $\pi_0(\mathrm{Diff}(\mathrm{T}_\mathrm{H}))$*:*

$$\Lambda_w = \{\lambda \in \mathbf{R} \mid \lambda\mathcal{M}_w = \mathcal{M}_w\} \quad \textit{with} \quad \mathcal{M}_w = w(\mathbf{Z}^n).$$

☞ Proof. The situation for the diffeomorphisms of the torus T_H is identical to the case of T_α. They are the projections f of the affine maps

$$F: x \mapsto \lambda x + \mu,$$

such that, for all $k \in \mathbf{Z}^n$ there exists a unique $k' \in \mathbf{Z}^n$ with $F(w(k)) = w(k')$. In other words,

$$\lambda w(\mathbf{Z}^n) \subset w(\mathbf{Z}^n).$$

The map $f \in \mathrm{Diff}(T_H)$ is the defined by

$$f \circ \mathrm{class}_w(x) = \mathrm{class}_w(F(x)).$$

On T_H, f is the composite of the linear part

$$\underline{\lambda}: \ \mathrm{class}_w(x) \mapsto \mathrm{class}_w(\lambda x)$$

by some translation

$$t_\rho: \ \mathrm{class}(x) \mapsto \mathrm{class}_w(x) + \rho \quad \text{with} \quad \rho = \mathrm{class}_w(\mu).$$

We can focus on the linear parts of the diffeomorphisms of T_H, which makes the discrete part of $\mathrm{Diff}(T_H)$.

Consider now the inverse diffeomorphism $(\underline{\lambda})^{-1}$, it can be lifted to $\mathbf{R}^n$ by $\underline{\lambda'}$, with

$$\underline{\lambda} \circ \mathrm{class}_w = \mathrm{class}_w \circ \underline{\lambda} \quad \text{and} \quad \mathrm{class}_w \circ \underline{\lambda'} = (\underline{\lambda})^{-1} \circ \mathrm{class}_w,$$

where $\underline{\lambda}$ on $\mathbf{R}$ is just the multiplication by λ. We get

$$\mathrm{class}_w \circ \underline{\lambda'} \circ \underline{\lambda} = \mathrm{class}_w,$$

which gives first $\lambda'\lambda x = x + w(k)$, with $k \in \mathbf{Z}^n$, and then $k = 0$ for $x = 0$. Therefore

$$\lambda' = \frac{1}{\lambda}.$$

Thus,

$$\frac{1}{\lambda} w(\mathbf{Z}^n) \subset w(\mathbf{Z}^n) \quad \Rightarrow \quad \lambda \times \frac{1}{\lambda} w(\mathbf{Z}^n) \subset \lambda w(\mathbf{Z}^n).$$

Therefore, $\lambda w(\mathbf{Z}^n) \subset w(\mathbf{Z}^n)$ and $w(\mathbf{Z}^n) \subset \lambda w(\mathbf{Z}^n)$, that is,

$$\lambda w(\mathbf{Z}^n) = w(\mathbf{Z}^n).$$

We get then the discrete part of $\mathrm{Diff}(T_H)$

$$\Lambda_w = \{\lambda \in \mathbf{R} \mid \lambda \mathcal{M}_w = \mathcal{M}_w\},$$

such that $\mathrm{Diff}(T_H) \simeq \Lambda_w \times T_H$. ▶

In order to understand the group of components Λ_w we will introduce the Q-vector space:

$$E_w = w(Q^n) = \Big\{ q_1 + \sum_{i=2}^{n} q_i w_i \mid q_i \in Q \Big\}.$$

<u>56. **The Algebraic Field** K_w.</u> *The set of numbers*

$$K_w = \{\lambda \in \mathbf{R} \mid \lambda E_w \subset E_w\}$$

is an algebraic number field, *a finite extension of* Q, *whose dimension* d *on* Q *divides* n, *and* E_w *is a* K_w*-vector space of dimension* n/d. *That is,* K_w *is a field* $Q(\theta)$ *where* θ *is a solution of some polynomial with integer coefficients.*

☞ Proof. It is enough to prove that if $k \in K_w$ and $k \neq 0$, then $1/k \in K$. The multiplication by k is a linear map in E_w whose kernel is $\{0\}$, then it is injective. Since E_w is finite dimensional, it is surjective: for all $y \in E_w$ there exists $x \in E$ such that $kx = y$, that is, $x = y/k$. the number $1/k$ stabizes E_w. On the other hand, K_w is a subalgebra of L(E), hence of finite dimension on Q. Since $Q \subset K_w$, K_w is a finite extension of Q. Moreover, the space E_w is naturally a K_w-module, it is then a K_w-vector space. We get then $\dim_Q E_w = \dim_Q K_w \times \dim_{K_w} E_w$. ▶

Let us consider now a lattice $\mathcal{M}_w \subset E_w$, that is, an additive subgroup of E_w such that $\mathcal{M}_w \otimes Q = E_w$. Its ring of stabilizers:

$$A_w = \{\lambda \in \mathbf{R} \mid \lambda \mathcal{M}_w \subset \mathcal{M}_w\}$$

is a subring of the field K_w.

Let us recall what is an *order* in the sense of ring theory (op. cit.).

<u>57. **Order of a ring.**</u> *Let* K *be a ring that is a <u>finite-dimensional algebra over the field</u>* <u>Q</u>. *Let* $A \subset K$ *be a subring. We say that* A *is an* order *of* K *if*

(1) A *is a* **Z**-*lattice in* K,
(2) A *spans* K *over* **Q**.

Then,

58. The Order $\mathcal{M}_w$. Let $E \subset \mathbf{R}$ be a finite dimensional **Q**-vector subspace, and $\mathcal{M} \subset E$ be a **Z**-lattice. The ring A of stabilizers of $\mathcal{M}$

$$A = \{\lambda \in \mathbf{R} \mid \lambda\mathcal{M} \subset \mathcal{M}\},$$

is an order of the ring K of the stabilizer of E in **R**

$$K = \{\lambda \in \mathbf{R} \mid \lambda E \subset E\}.$$

In other words:

$$E = \mathcal{M} \otimes \mathbf{Q} \quad \Rightarrow \quad K = A \otimes \mathbf{Q}.$$

Proof. We want to prove that $K = A \otimes \mathbf{Q}$. Let $w = (w_1, \ldots, w_n)$ be a **Z**-basis of $\mathcal{M}$, *i.e.* a **Q**-basis of $E = \mathcal{M} \otimes \mathbf{Q}$ such that $w(\mathbf{Z}^n) = \mathcal{M}$. Let $\lambda \in K$ and Λ be the matrix representing $\underline{\lambda}$, the multiplication by $\lambda \in K$, in the basis w. The matrix Λ can be written $\Lambda = \Lambda'/\ell$, where $\ell \in \mathbf{Z}$ is the least common multiple of the denominators of the elements of Λ, and $\Lambda' \in L(\mathbf{Z}^n)$. For all $m \in \mathcal{M}$ we have then $\ell\lambda m \in \mathcal{M}$, that is, $(\ell\lambda)\mathcal{M} \subset \mathcal{M}$. Therefore, $\ell\lambda \in A$, or again $\lambda \in A \otimes \mathbf{Q}$. ▶

So, coming back to A_w and K_w, A_w is an order of K_w. Now we are not just interested in A_w but in its invertible elements. That is,

$$\begin{aligned} \Lambda_w &= \{\lambda \in \mathbf{R} \mid \lambda\mathcal{M} \subset \mathcal{M} \text{ and } \frac{1}{\lambda}\mathcal{M} \subset \mathcal{M}\}, \\ &= \{\lambda \in A_w \mid \lambda^{-1} \in A_w\}. \end{aligned}$$

59. The Group Λ_w. The group Λ_w of components of $\mathrm{Diff}(T_H))$ is the group of invertible elements of the ring A_w, that is, its group of units (op. cit.). Since A_w is an order of the algebraic field K_w, its group of units is given by the *Dirichlet's unit theorem.*[1] In our case:

$$\Lambda_w \simeq \pm 1 \times \mathbf{Z}^{r+s-1},$$

[1] https://www.wikipedia.org/en/Dirichlet's_unit_theorem

where r is the number of real places of the field K_w and $2s$ the number of complex places. In other words, $K_w = \mathbf{Q}(\theta)$ where θ is a solution of a polynomial P with integer coefficients. The degree d of K_w divides n, thus $d = r + 2s$ and $n = \ell d$.

Note. In particular, for $n = 2$ there are two cases, either $d = 0$ and $\Lambda_w = \{\pm 1\}$, or $d = 2$ and $\Lambda_w = \{\pm 1\} \times \mathbf{Z}$.

60. Example. For the quotient of a 2-torus there is, as we have seen, only one possibility: if we have one real root of the characteristic polynomial, we have two and then $\pi_0(\mathrm{Diff}(T/H) = \{\pm 1\} \times \mathbf{Z}$. For a 3-torus we have two possibilities either our characteristic polynomial has one real root or three, illustrated in Figure 13.

In the first case the order is generated by one generator χ, and $\pi_0(\mathrm{Diff}(T/H) = \{\pm 1\} \times \mathbf{Z}$.

$$w = (1 \ \sqrt[3]{2}), \quad \chi = \sqrt[3]{2} - 1.$$

In the second case the order of K_w has two generators χ and χ', and $\pi_0(\mathrm{Diff}(T/H') = \{\pm 1\} \times \mathbf{Z}^2$.

$$w = (1 \ \rho \ \rho^2), \text{ with } \rho = 2\cos\left(\frac{2\pi}{9}\right), \ \chi = \rho, \text{ and } \chi' = \frac{1}{1-\rho}.$$

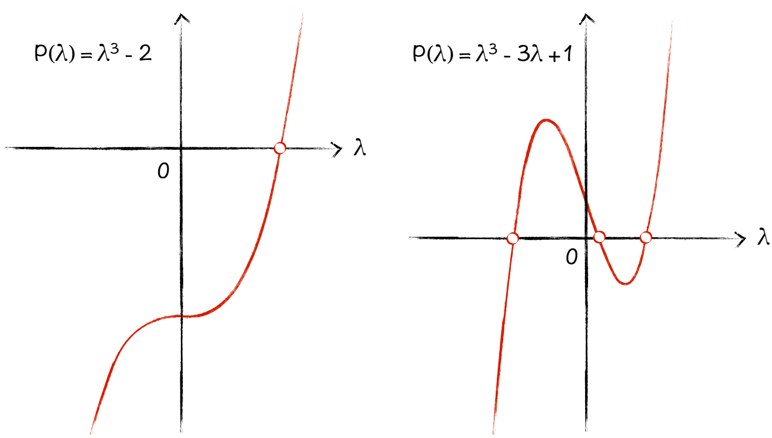


Figure 13. The two examples for T^3.

Generating Families, Dimension

In this lecture we introduce first the notion of *generating family*, and we'll see an application in the definition of *dimension in diffeology*. Applied to the quotients $\Delta_m = \mathbf{R}^m/\mathrm{O}(\mathbf{R}^m)$, we prove that and Δ_m and Δ_n are not diffeomorphic if $n \neq m$, and not diffeomorphic to the half-line $\Delta_\infty = [0, \infty[\subset \mathbf{R}$.

The notion of *dimension* in diffeology, that was introduced in **[PIZ07]**, is a quick and easy answer to the question: *For two different integers n and m, are the diffeological spaces* $\Delta_n = \mathbf{R}^n/\mathrm{O}(n)$ *and* $\Delta_m = \mathbf{R}^m/\mathrm{O}(m)$ *diffeomorphic?* We will show that since $\dim(\Delta_n) = n$ and since the dimension is a diffeological invariant, the answer is *No, they are not*. This method simplifies a partial result, obtained in a more complicated way in **[Igl85]**, stating that Δ_1 and Δ_2 are not diffeomorphic. The half-line $\Delta_\infty = [0, \infty[\subset \mathbf{R}$ is a similar example for which $\dim(\Delta_\infty) = \infty$. Hence, Δ_m is not diffeomorphic to the half-line Δ_∞ for any integer m. Dimension is a simple but powerful diffeological invariant.

18. Generating famillies

Diffeologies can be built by *generating families*. Any family of parametrizations of a set generates a diffeology. Conversely, any diffeology is generated by some set of parametrizations. This mode of construction of diffeologies is very useful because it can reduce the analysis of the properties of a diffeological space to a subset of its plots, hopefully smaller than the whole diffeology. The definition of generating diffeology leads to the definition of the dimension of a diffeological space, which is a first global invariant of the category {Diffeology}. And this construction also leads to the introduction of important subcategories of diffeological spaces, for example the category of manifolds, or the category of orbifolds and others.

61. Proposition. Let $\mathcal{F}$ be a familly of parametrizations of a set X. There exists a finest diffeology on X containing $\mathcal{F}$, it is called the *diffeology generated by* $\mathcal{F}$. It is denoted by $\langle \mathcal{F} \rangle$. The family $\mathcal{F}$ is said to be a *generating family* of the diffeological space $(X, \langle \mathcal{F} \rangle)$ **[PIZ07]**. It is the intersection of the diffeologies containing $\mathcal{F}$

$$\langle \mathcal{F} \rangle = \bigcap_{\substack{\mathcal{D} \in \mathrm{Diffeologies}(X) \\ \mathcal{F} \subset \mathcal{D}}} \mathcal{D}.$$

Let X be a diffeological space, the set of families generating X will be denoted by Gen(X).

If the family $\mathcal{F}$ covers X, that is, if for all $x \in X$ there exists a parametrization $F \in \mathcal{F}$ such that $x = F(r)$, with $r \in \mathrm{dom}(F)$, then a plot of $\langle \mathcal{F} \rangle$ is any parametrization $P : U \to X$ such that:

(♣) For all $r \in U$ there exists an open neighborhood $V \subset U$ of r, a parametrization $F \in \mathcal{F}$, and a smooth parametrization $Q : V \to \mathrm{dom}(F)$ such that $P \restriction V = F \circ Q$.

If the family $\mathcal{F}$ does not cover X, the easiest way is to add the constant parametrizations

$$\hat{x} : \mathbf{R}^0 \to X \quad \text{with} \quad \hat{x}(0) = x,$$

to $\mathcal{F}$. That gives a family $\hat{\mathcal{F}}$ that covers X and

$$\langle \mathcal{F} \rangle = \langle \hat{\mathcal{F}} \rangle.$$

Note 1. The empty family $\varnothing$ generates the discrete family,

$$\langle \varnothing \rangle = \mathcal{D}_\circ.$$

Note 2. The diffeology $\mathcal{D}$ of a diffeological space X belongs to Gen(X). Generating family is a projector:

$$\langle \mathcal{D} \rangle = \mathcal{D}$$

Note 3. Generating families is an increasing function of fineness

$$\mathcal{F} \subset \mathcal{F}' \quad \Rightarrow \quad \langle \mathcal{F} \rangle \subset \langle \mathcal{F}' \rangle.$$

62. Pushing forward families. Let X and X$'$ be two sets. Let $\mathcal{F}$ be a family of parametrizations of X, and let $f : X \to X'$ be a map.

The pushforward $f_*(\mathcal{F})$ of the family $\mathcal{F}$ by f is defined by

$$f_*(\mathcal{F}) = \left\{ f \circ F \mid F \in \mathcal{F} \right\}.$$

Then, the diffeology generated by the pushforward of the family $\mathcal{F}$ by f is the pushforward by f of the diffeology generated by $\mathcal{F}$, that is,

$$\langle f_*(\mathcal{F}) \rangle = f_*(\langle \mathcal{F} \rangle).$$

In particular, let X and X$'$ be two diffeological spaces, and let $f : X \to X'$ be a subduction. The pushforward $f_*(\mathcal{F})$ of any generating family $\mathcal{F}$ for X is a generating family for X$'$.

63. Pulling back families. Let X and X$'$ be two sets. Let $\mathcal{F}'$ be a family of parametrizations of X$'$, and let $f : X \to X'$ be any map. Let us define the *pullback of the family* $\mathcal{F}'$ *by* f as the family $f^*(\mathcal{F})$ of parametrizations $F : U \to X$ satisfying the following property:

- (♦) Either $f \circ F$ is constant or there exists an element $F' : U' \to X'$ of $\mathcal{F}'$, and a smooth parametrization $\phi : U \to U'$, such that $F' \circ \phi = f \circ F$.

Then, the diffeology generated by the pullback $f^*(\mathcal{F}')$ is the pullback by f of the diffeology generated by $\mathcal{F}$, that is,

$$\langle f^*(\mathcal{F}') \rangle = f^*(\langle \mathcal{F}' \rangle).$$

In particular, let X and X$'$ be two diffeological spaces, and let $f : X \to X'$ be an induction. The pullback $f^*(\mathcal{F}')$ of any generating family $\mathcal{F}'$ for X$'$ is a generating family for X. Unfortunately, compared with the pushforward of a family, pulling back a small generating family may lead to a huge family, almost as big as the diffeology itself. That will be the next example.

Note. The choice of a generating family is relatively arbitrary. For example, the empty family is equivalent to the family of constant parametrizations. If the family $\mathcal{F}'$ is empty, its pullback is not empty, but is the set of the parametrizations of X with values in the preimages of points $f^{-1}(x')$, $x' \in X'$. This is not surprising since the pullback of the discrete diffeology is the sum of the preimages of points, equipped with the coarse diffeology.

64. Example: Generating the half-line. Let the half line $[0,\infty[\subset \mathbf{R}$ be equipped with the subset diffeology of $\mathbf{R}$. Let $\mathcal{F}$ be the generating family of $\mathbf{R}$ reduced to the identity, $\mathcal{F} = \{1_{\mathbf{R}}\}$. Then the pullback of the generating family $\mathcal{F}$ by the inclusion $j\colon [0,\infty[\to \mathbf{R}$ is the whole diffeology of $[0,\infty[$.

Proof. Indeed, the pullback $j^*(\{1_{\mathbf{R}}\})$ is the set of parametrizations $F\colon U \to [0,\infty[$ such that $j \circ F$ is constant, or there exists an element $F' \in \{1_{\mathbf{R}}\}$ and a smooth parametrization $\phi : U \to \mathrm{dom}(F')$ such that $j \circ F = F' \circ \phi$.

Thus, $F' = 1_{\mathbf{R}}$, $\mathrm{dom}(F') = \mathbf{R}$, and then $F = \phi$. Therefore, F is any smooth parametrization of $\mathbf{R}$ with values in $[0,\infty[$, and $j^*(\{1_{\mathbf{R}}\})$ is the whole diffeology of the half-line. ▶

19. Dimension of a diffeology

The *global dimension* of a diffeology, or a diffeological space, defined now is a diffeological invariant. It will be refined later in a pointwise dimension map.

65. Dimension of a parametrization. Let P be a n-parametrization of some set X, $n \in \mathrm{N}$. We say that n is the *dimension* of P, and we denote it by dim(P). That is,

$$\forall P \in \mathrm{Param}(X),\ \dim(P) = n \ \Leftrightarrow\ P \in \mathrm{Param}_n(X).$$

66. Dimension of a family of parametrizations. Let X be a set and let $\mathcal{F}$ be any family of parametrizations of X. We define the *dimension* of $\mathcal{F}$ as the supremum of the dimensions of its elements,

$$\dim(\mathcal{F}) = \sup\{\dim(F) \mid F \in \mathcal{F}\}.$$

Note that $\dim(\mathcal{F})$ can be infinite if for all $n \in \mathbf{N}$ there exists an element F of $\mathcal{F}$ such that $\dim(F) = n$. In this case we denote $\dim(\mathcal{F}) = \infty$.

67. Dimension of a diffeology. Let $\mathcal{D}$ be a diffeology. The *dimension* of $\mathcal{D}$ is defined as the infimum of the dimensions of its generating families:

$$\dim(\mathcal{D}) = \inf\{\dim(\mathcal{F}) \mid \langle \mathcal{F} \rangle = \mathcal{D}\}.$$

Let X be a diffeological space and $\mathcal{D}$ be its diffeology, we define the dimension of X as the dimension of $\mathcal{D}$:

$$\dim(X) = \dim(\mathcal{D}) \in \mathbf{N} \cup \{\infty\}.$$

68. Dimensions of Euclidean Domains. The diffeological dimension of an Euclidean domain $U \subset \mathbf{R}^n$, equipped with the standard diffeology, is equal to n:

$$\forall U \in \mathrm{Domains}(\mathbf{R}^n), \quad \dim(U) = n.$$

☞ Proof. Let 1_U be the identity map of U. The singleton $\{1_U\}$ is a generating family of U, therefore, $\dim(U) \leq \dim\{1_U\}$. Since $\dim\{1_U\} = n$, $\dim(U) \leq n$. Now let us assume that $\dim(U) < n$. Then, there exists a generating family $\mathcal{F}$ for U such that $\dim(\mathcal{F}) < n$. Since the identity map 1_U is a plot in U, it lifts locally at every point along some element of $\mathcal{F}$. Thus, for any $r \in U$ there exists a superset V of r, a parametrization $F\colon W \to U$, element of $\mathcal{F}$ (that is, $F \in \mathcal{C}^\infty(W, U)$) and a smooth map $Q\colon V \to W$, such that $1_U \restriction V = 1_V = F \circ Q$. But $\dim(\mathcal{F}) < n$ implies that $\dim(F) = \dim(W) < n$. Now, the rank of the linear tangent map $D(F \circ Q)$ is less or equal to $\dim(W) < n$, but $D(F \circ Q) = D(1_V) = 1_{\mathbf{R}^n}$, thus $\mathrm{rank}(D(F \circ Q)) = \mathrm{rank}(1_{\mathbf{R}^n}) = n$. Therefore, there is no generating family $\mathcal{F}$ of U with $\dim(\mathcal{F}) < n$, and $\dim(U) = n$. ▶

69. Dimension zero spaces are discrete. A diffeological space has dimension zero if and only if it is discrete.

☞ Proof. Let X be a set equipped with the discrete diffeology. Any plot $P: U \to X$ is locally constant. Then, for any $r \in U$, P lifts locally along the 0-plot $\mathbf{x} = [0 \mapsto x]$, where $x = P(r)$. Hence, the 0-plots form a generating family and $\dim(X) = 0$. Conversely, let X be a diffeological space such that $\dim(X) = 0$. Then, the 0-plots generate the diffeology of X. But, any plot lifting locally along a 0-plot is locally constant. Therefore, X is discrete. ▶

70. The dimension is a diffeological invariant. If two diffeological spaces are diffeomorphic, then they have the same dimension.

☞ Proof. Let X and X′ be two diffeological spaces and let $f \in \mathrm{Diff}(X, X')$. Let $\mathcal{F}$ be a generating family of X. The pushforward $\mathcal{F}' = f_*(\mathcal{F})$ made of the plots $f \circ F$, where $F \in \mathcal{F}$, is clearly a generating family of X′. Conversely we have f^{-1}. Therefore, $\dim(X) = \dim(X')$. ▶

71. Exemple: Has the set $\{0, 1\}$ dimension 1? Let us realise the set $\{0, 1\}$ as the quotient of the real line $\mathbf{R}$ by the projection

$$\begin{cases} \pi: \mathbf{R} \to \{0, 1\} \\ \pi(x) = 0 \text{ if } x \in \mathbf{Q}, \text{ and } \pi(x) = 1 \text{ otherwise.} \end{cases}$$

Let then $\{0, 1\}_\pi$ be the set $\{0, 1\}$ equipped with the diffeology generated by the parametrization π. Since $\{\pi\}$ is a generating family, the dimension of $\{0, 1\}_\pi$ is less than or equal to $1 = \dim(\{\pi\})$. But, since the plot π is not locally constant, by density of the rational (or irrational) numbers in $\mathbf{R}$, the space $\{0, 1\}_\pi$ is not discrete. Hence, $\dim\{0, 1\}_\pi \neq 0$, and finally $\dim\{0, 1\}_\pi = 1$. Thus, a finite diffeological space may have a dimension non zero.

72. Exemple: Dimension of tori. Let $\Gamma \subset \mathbf{R}$ be any strict subgroup of $(\mathbf{R}, +)$ and let T_Γ be the quotient $\mathbf{R}/\Gamma$, whose diffeology is generated by the projection class: $\mathbf{R} \to \mathbf{R}/\Gamma$. Then,

$$\dim(T_\Gamma) = 1.$$

This applies in particular to the circles $\mathbf{R}/a\mathbf{Z}$, with perimeter $a > 0$, or to *irrational tori* when Γ is generated by more than one generators, rationally independent.

☞ Proof. Since $\mathbf{R}$ is an Euclidean domain, class is a plot of the quotient, and $\mathcal{F} = \{\text{class}\}$ is a generating family of $\mathbf{R}/\Gamma$, and $\dim(\mathcal{F}) = 1$. Thus, as a direct consequence of the definition $\dim(\mathbf{R}/\Gamma) \leq 1$. Now, if $\dim(\mathbf{R}/\Gamma) = 0$, then the diffeology of the quotient is generated by the constant parametrizations. But π is not locally constant, therefore $\dim(\mathbf{R}/\Gamma) = 1$. ▶

20. Dimension map of a diffeological space

Because diffeological spaces are not necessarily homogeneous, the global dimension of a diffeological space is a too rough invariant. It is necessary to refine this definition and to introduce the dimension function of a diffeological space, defined at each of its points.

The dimension function of diffeological spaces is the simplest numeral invariant in diffeology.

73. Pointed plots and germ of a diffeological space. Let X be a diffeological space, let $x \in X$. Let $P: U \to X$ be a plot. We say that P is *pointed at x* if $0 \in U$ and $P(0) = x$. We will agree that the set of *germs* of the pointed plots of X at x represents the *germ of the diffeology* at this point, and we shall denote it by $\mathcal{D}_x$.

74. Local generating families. Let X be a diffeological space and let x be a point in X. We shall call *local generating family at x* any family $\mathcal{F}$ of plots of X such that:

(1) Every element P of $\mathcal{F}$ is pointed at x, that is, $0 \in \operatorname{dom}(P)$ and $P(0) = x$.
(2) For all plots $P: U \to X$ pointed at x, there exists a superset V of $0 \in U$, a parametrization $F: W \to X$ belonging to $\mathcal{F}$ and a smooth parametrization $Q: V \to W$ pointed at $0 \in W$, such that $F \circ Q = P \upharpoonright V$.

We shall say also that $\mathcal{F}$ *generates the germ* $\mathcal{D}_x$ of the diffeology $\mathcal{D}$ of X at the point x. And we denote

$$\mathcal{D}_x = \langle \mathcal{F} \rangle.$$

Note that, for any x in X, the set of local generating families at x is not empty, since it contains the set of all the plots pointed at x, and this set contains the constant parametrizations with value x.

75. Local generating families. Let X be a diffeological space. Let us choose, for every $x \in X$, a local generating family $\mathcal{F}_x$ at x. The union $\mathcal{F}$ of all these local generating families,

$$\mathcal{F} = \bigcup_{x \in X} \mathcal{F}_x,$$

is a generating family of the diffeology of X.

Proof. Let $P \colon U \to X$ be a plot, let $r \in U$ and $x = P(r)$. Let T_r be the translation $T_r(r') = r' + r$. Let $P' = P \circ T_r$ defined on $U' = T_r^{-1}(U)$. Since the translations are smooth, the parametrization P' is a plot of X. Moreover P' is pointed at x, $P'(0) = P \circ T_r(0) = P(r) = x$. By definition of a local generating family, there exists an element $F \colon W \to X$ of $\mathcal{F}_x$, a superset V' of $0 \in U'$ and a smooth parametrization $Q' : V' \to W$, pointed at 0, such that $P' \upharpoonright V' = F \circ Q'$. Thus, $P \circ T_r \upharpoonright V' = F \circ Q'$, that is, $P \upharpoonright V = F \circ Q$, where $V = T_r(V')$ and $Q = Q' \circ T_r^{-1}$. Hence, P lifts locally, at every point of its domain, along an element of $\mathcal{F}$. Therefore, $\mathcal{F}$ is a generating family of the diffeology of X. ▶

76. The dimension map. Let X be a diffeological space and let x be a point of X. By analogy with the global dimension of X, we define the *dimension of* X *at the point* x by:

$$\dim_x(X) = \inf\{\dim(\mathcal{F}) \mid \langle \mathcal{F} \rangle = \mathcal{D}_x\}.$$

The map $x \mapsto \dim_x(X)$, with values in $\mathbf{N} \cup \{\infty\}$, is called the *dimension map* of the space X.

77. Global dimension and dimension map. Let X be a diffeological space. The global dimension of X is the supremum of the dimension map of X:

$$\dim(\mathrm{X}) = \sup_{x\in \mathrm{X}} \left\{\dim_x(\mathrm{X})\right\}.$$

☞ Proof. Let $\mathcal{D}$ be the diffeology of X. Let us prove first that for every $x \in \mathrm{X}$, $\dim_x(\mathrm{X}) \leq \dim(\mathrm{X})$, which implies that $\sup_{x\in \mathrm{X}} \dim_x(\mathrm{X}) \leq \dim(\mathrm{X})$. For that we shall prove that for any $x \in \mathrm{X}$ and any generating family $\mathcal{F}$ of $\mathcal{D}$, $\dim_x(\mathrm{X}) \leq \dim(\mathcal{F})$. Then, since $\dim(\mathrm{X}) = \inf\{\dim(\mathcal{F}) \mid \mathcal{F} \in \mathcal{D} \text{ and } \langle\mathcal{F}\rangle = \mathcal{D}\}$ we shall get, $\dim_x(\mathrm{X}) \leq \dim(\mathrm{X})$.

Now, let $\mathcal{F}$ be a generating family of $\mathcal{D}$. For any plot $\mathrm{P}\colon \mathrm{U} \to \mathrm{X}$ pointed at x, let us choose an element F of $\mathcal{F}$ such that: there exists a superset V of $0 \in \mathrm{U}$ and a smooth parametrization $\mathrm{Q} : \mathrm{V} \to \mathrm{def}(\mathrm{F})$, such that $\mathrm{F} \circ \mathrm{Q} = \mathrm{P} \restriction \mathrm{V}$.

Then, let $r = \mathrm{Q}(0)$ and T_r be the translation $\mathrm{T}_r(r') = r' + r$. Let $\mathrm{F}' = \mathrm{F} \circ \mathrm{T}_r$, defined on $\mathrm{T}_r^{-1}(\mathrm{def}(\mathrm{F}))$.

Thus, $\mathrm{F}'(0) = x$, and F' is a plot of X, pointed at x, such that $\dim(\mathrm{F}') = \dim(\mathrm{F})$. Let $\mathrm{Q}' = \mathrm{T}_r^{-1} \circ \mathrm{Q}$, then Q' is smooth and $\mathrm{P} \restriction \mathrm{V} = \mathrm{F}' \circ \mathrm{Q}'$.

Thus, the set $\mathcal{F}'_x$ of all these plots F' associated with the plots pointed at x is a generating family of $\mathcal{D}_x$, and for each of them $\dim(\mathrm{F}') = \dim(\mathrm{F}) \leq \dim(\mathcal{F})$.

Hence, $\dim(\mathcal{F}'_x) \leq \dim(\mathcal{F})$. But $\dim_x(\mathrm{X}) \leq \dim(\mathcal{F}'_x)$, thus $\dim_x(\mathrm{X}) \leq \dim(\mathcal{F})$. And we conclude that $\dim_x(\mathrm{X}) \leq \dim(\mathrm{X})$, for any $x \in \mathrm{X}$, and $\sup_{x\in \mathrm{X}} \dim_x(\mathrm{X}) \leq \dim(\mathrm{X})$.

Next, let us prove that $\dim(\mathrm{X}) \leq \sup_{x\in \mathrm{X}} \dim_x(\mathrm{X})$. Let us assume that $\sup_{x\in \mathrm{X}} \dim_x(\mathrm{X})$ is finite. Otherwise, according to the previous part we have $\sup_{x\in \mathrm{X}} \dim_x(\mathrm{X}) \leq \dim(\mathrm{X})$, and then $\dim(\mathrm{X})$ is infinite and $\sup_{x\in \mathrm{X}} \dim_x(\mathrm{X}) = \dim(\mathrm{X})$.

Now, for any $x \in \mathrm{X}$, $\dim_x(\mathrm{X})$ is finite. And since the sequence of the dimensions of the generating families of $\mathcal{D}_x$ is lower bounded, there exists for any x a generating family $\mathcal{F}_x$ such that $\dim_x(\mathrm{X}) = \dim(\mathcal{F}_x)$. For every x in X let us choose one of them.

Next, let us define $\mathcal{F}_m$ as the union of all these chosen families. According to a proposition above, $\mathcal{F}_m$ is a generating family of $\mathcal{D}$.

Hence, $\dim(X) \leq \dim(\mathcal{F}_m)$. But $\dim(\mathcal{F}_m) = \sup_{F\in\mathcal{F}_m} \dim(F) = \sup_{x\in X} \sup_{F\in\mathcal{F}_x} \dim(F) = \sup_{x\in X} \dim(\mathcal{F}_x) = \sup_{x\in X} \dim_x(X)$.

Therefore, $\dim(X) \leq \sup_{x\in X} \dim_x(X)$. And we can conclude, from the two parts above, that $\dim(X) = \sup_{x\in X} \dim_x(X)$. ▶

<u>78. The dimension map is a local invariant.</u> Let X and X′ be two diffeological spaces. If $x \in X$ and $x' \in X'$ are two points related by a local (a fortiori global) diffeomorphism, then $\dim_x(X) = \dim_{x'}(X')$.

<u>79. Dimensions of manifolds.</u> A n-manifold is a diffeological space X generated by local diffeomorphisms with $\mathbf{R}^n$. Each such local diffeomorphism is called a *chart* of X. Note that, each point is locally equivalent to any other point: there exists always a local diffeomorphism from one point to any other. The dimension function is constant, equal to the global dimension. Now, since there is always a chart mapping $0 \in \mathbf{R}^n$ to $x \in X$, $\dim_x(X) = \dim_0(\mathbf{R}^n) = n$. Therefore, $\dim(X) = n$. Which is coherent with the ususal dimension in differential geometry.

<u>80. Klein decomposition.</u> Let X be a diffeological space. The local diffeomorphisms of X split the space into classes, according to the relation: $x \sim x'$ if and only if there exists a local diffeomorphism f mapping x to x'. These classes are called the *orbits of the local diffeomorphisms* of X. We call these classes *Klein's orbit*. The dimension map is constant on each Klein's orbit.

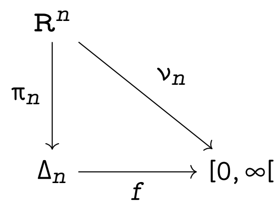
21. Examples of the half-lines

<u>81. The half-line Δ_n.</u> Let $\Delta_n = \mathbf{R}^n/\mathrm{O}(n)$ equipped with the quotient diffeology, $n \in \mathbf{N}$. Then, $\dim_0(\Delta_n) = n$, and $\dim_x(\Delta_n) = 1$ if $x \neq 0$. Thus, $\dim(\Delta_n) = n$ and for $n \neq m$ the half-lines Δ_n and Δ_m are not diffeomorphic.

Proof. Let $n > 0$, and let us denote by $\mathrm{class}_n : \mathbf{R}^n \to \Delta_n$ the projection from $\mathbf{R}^n$ onto its quotient. There is a natural bijection $f : \Delta_n \to [0, \infty[$ such that

$$f \circ \mathrm{class}_n = \nu_n \quad \text{with} \quad \nu_n(x) = \|x\|^2.$$

Now, thanks to the uniqueness of quotients, we use f to identify Δ_n with $[0, \infty[$, equipped with the diffeology $\mathcal{D}_n$ generated by ν_n.



The elements of $\mathcal{D}_n$ consist of the parametrizations which can be lifted locally along ν_n by smooth parametrizations of $\mathbf{R}^n$. Thus, since $\dim(\nu_n) = n$, we get $\dim(\Delta_n) \leq n$. Let us prove now that $\dim(\Delta_n) \geq n$:

Let us assume that ν_n, which is an element of $\mathcal{D}_n$, can be lifted locally, at the point 0_n, along a plot

$$P \in \mathcal{D}_n \quad \text{with} \quad \dim(P) = p < n.$$

Then, there exists a smooth parametrization

$$\phi : V \to \operatorname{dom}(P) \quad \text{such that} \quad P \circ \phi = \nu_n \restriction V.$$

We can assume without loss of generality that

$$0_p \in \operatorname{dom}(P),\ P(0_p) = 0 \ \text{ and } \ \phi(0_n) = 0_p.$$

Now, since P is an element of $\mathcal{D}_n$, there exists a smooth parametrization

$$\psi\colon W \to \mathbf{R}^n \ \text{ such that } \ 0_p \in W \ \text{ and } \ \nu_n \circ \psi = P \restriction W.$$

Let $V' = \phi^{-1}(W)$, we get

$$\nu_n \restriction V' = \nu_n \circ F \ \text{ with } \ F = \psi \circ \phi \restriction V',$$

and

$$F \in \mathcal{C}^\infty(V', \mathbf{R}^n),\ 0_n \in V' \ \text{ and } \ F(0_n) = 0_n,$$

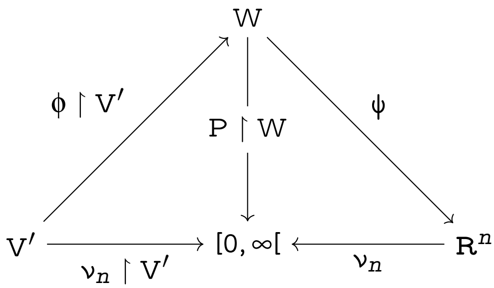
that is

$$\|x\|^2 = \|F(x)\|^2.$$

The second derivative of this identity computed at the point 0_n gives

$$\mathbf{1}_n = M^t M \quad \text{with} \quad M = D(F)(0_n).$$

This is summarized by the following diagram:



But

$$M = AB \quad \text{with} \quad A = D(\psi)(0_p) \quad \text{and} \quad B = D(\phi)(0_n).$$

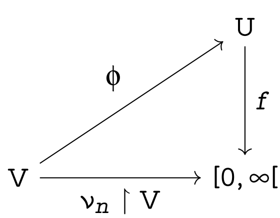
Thus, $1_n = B^t A^t AB$, which is impossible because $\operatorname{rank}(B) \leq p < n$. Therefore, $\dim(\Delta_n) = n$. And, since the dimension is a diffeological invariant, Δ_n is not diffeomorphic to Δ_m for $n \neq m$. ▶

82. The half-line Δ_∞. The dimension of a diffeological subspace $A \subset X$ can be less, equal, or even greater than the dimension of X. The following example is an illustration of this phenomenon. Let $\Delta_\infty = [0, \infty[\subset \mathbf{R}$, equipped with the subset diffeology. Then,

$$\dim_0(\Delta_\infty) = \infty \quad \text{and} \quad \dim_x(\Delta_\infty) = 1 \quad \text{if} \quad x \neq 0.$$

Thus, $\dim(\Delta_\infty) = \infty$, and for any integer m, Δ_∞ is not diffeomorphic to Δ_m.

☞ Proof. Let us assume that $\dim(\Delta_\infty) = N < \infty$. For any integer n, the map $\nu_n : \mathbf{R}^n \to \Delta_\infty$, defined by $\nu_n(x) = \|x\|^2$, belongs to $\mathcal{D}_\infty$, the subset diffeology on $[0, \infty[$.



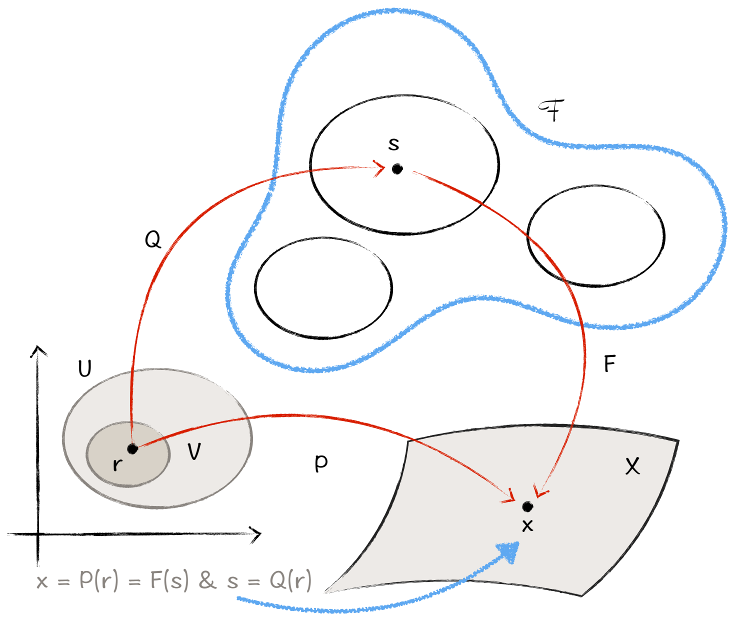


Figure 14. A generating family.

Hence, ν_n lifts locally at the point 0_n along some $P \in \mathcal{D}_\infty$, where $\dim(P) = p \leq N$. Now, let us choose $n > N$. Then, P belongs to some $\mathcal{C}^\infty(U, \mathbf{R})$ with $\mathrm{val}(P) \subset [0, \infty[$, and there exists a smooth parametrization $\phi : V \to U$ such that $P \circ \phi = \nu_n \restriction V$. We can assume, without loss of generality, that $0_p \in U$, $\phi(0_n) = 0_p$, and thus: $P(0_p) = 0$.

Now, the first derivative of ν_n at a point $x \in V' = \phi^{-1}(V)$ is given by $x = D(P)(\phi(x)) \circ D(\phi)(x)$. But, since P is smooth and positive, and since $P(0) = 0$ we have $D(P)(0_p) = 0$.

Hence, the second derivative of ν_n computed at the point 0_n gives $\mathbf{1}_n = M^t H M$, where $M = D(\phi)(0)$ and $H = D^2(P)(0)$. But since $\mathrm{rank}(M) \leq p \leq N$ and $n > N$, this is impossible, and the dimension of the embedded line is infinite at the origin, $\dim(\Delta_\infty) = \infty$. ▶

Cartan-De-Rham Calculus

This lecture is about the crucial theory of differential calculus, also called Cartan calculus. It is accompanied with the description of the De Rham Cohomology and its principal properties.

In this lecture we shall see the elementary constructions and definitions of the theory of differential calculus in diffeology. In particular, a reminder about smooth forms in Euclidean spaces, the definitions of differential forms in diffeology, the operations of pullbacks and exterior differential, the behavior under subduction with an example on orbifold, the definition of the De Rham cohomology, the definition of the cubic homology and the integration of forms on chains, the De Rham homomorphism.

22. Smooth forms in Euclidean spaces

83. Linear p-forms on $\mathbf{R}^n$. A *linear p-form* on the Euclidean space $\mathbf{R}^n$ is a map

$$\alpha\colon (\mathbf{R}^n)^p \to \mathbf{R}$$

which is *multilinear*, that is, that is linear in each parameter,

$$\begin{aligned}\alpha(v_1,\ldots,\lambda v_i+\mu v_i',\ldots,v_p) &= \lambda\alpha(v_1,\ldots,v_i,\ldots,v_p)\\ &+ \mu\alpha(v_1,\ldots,v_i',\ldots,v_p)\end{aligned}$$

for all $i = 1 \dots p$, and which is antisymmetric under every *transposition*,

$$\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_p) = -\alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_p).$$

We deduce from the antisymmetry that for all *permutation* ε of $\{1, \dots, p\}$

$$\alpha(v_{\varepsilon(1)}, \dots, v_{\varepsilon(p)}) = (-1)^{\mathrm{sgn}(\varepsilon)} \alpha(v_1, \dots, v_p),$$

where sgn denotes the *signature* of the permutations.

We denote

$$\Lambda^p(\mathbf{R}^n)$$

the set of p-forms (a shortcut for linear p-forms). This set is naturally a real vector space:

$$(\lambda\alpha + \mu\beta)(v_1, \dots, v_p) = \lambda\alpha(v_1, \dots, v_p) + \mu\beta(v_1, \dots, v_p).$$

Note that:

$$\Lambda^0(\mathbf{R}^n) = \mathbf{R} \quad \text{and} \quad \Lambda^1(\mathbf{R}^n) = (\mathbf{R}^n)^* \simeq \mathbf{R}^n,$$

the *dual space* of $\mathbf{R}^n$. The dimension is the binomial coefficient

$$\dim(\Lambda^p(\mathbf{R}^n)) = \frac{n!}{p!\ (n-p)!}.$$

Note that:

$$\dim(\Lambda^n(\mathbf{R}^n)) = 1 \quad \text{and} \quad \dim(\Lambda^p(\mathbf{R}^n)) = 0 \quad \text{if } p > n.$$

Note. It happens that I write

$$\alpha(v_1) \cdots (v_p) \quad \text{for} \quad \alpha(v_1, \cdots, v_p).$$

84. Pullbacks of linear forms. Consider a linear map

$$\mathrm{M}\colon \mathbf{R}^n \to \mathbf{R}^m$$

Let $\beta \in \Lambda^p(\mathbf{R}^m)$, We call *pullback* of β by M the map denoted and defined by

$$\mathrm{M}^*(\beta)(v_1, \dots, v_p) = \beta(\mathrm{M}v_1, \dots, \mathrm{M}v_p).$$

The map $M^*(\beta)$ is obviously a linear p-form on $\mathbf{R}^n$. Moreover, M^* is linear:

$$M^* \in L\Big(\Lambda^p(\mathbf{R}^m), \Lambda^p(\mathbf{R}^n)\Big).$$

<u>**85. Smooth forms on Euclidean domains.**</u> Let $U \subset \mathbf{R}^n$, we call *smooth p-form* on U any smooth map

$$\alpha \colon U \to \Lambda^p(\mathbf{R}^n).$$

We denote

$$\Omega^p(U) = \mathcal{C}^\infty\Big(U, \Lambda^p(\mathbf{R}^n)\Big)$$

the space of smooth p-form on U. In this case: $\alpha \in \Omega^p(U)$, $\alpha(x) \in \Lambda^p(\mathbf{R}^n)$ for all $x \in U$.

There are a few ways of defining "smooth" for a p-form, one can use the canonical decomposition of a linear p-form on the canonical basis (it is documented everywhere, and also in [TB]), or we can directly define it by this property:

$$\text{For all } v_1, \ldots, v_p \in \mathbf{R}^n \quad [x \mapsto \alpha(x)(v_1, \ldots, v_p)] \in \mathcal{C}^\infty(U, \mathbf{R}).$$

The expression $\alpha(x)(v_1, \ldots, v_p)$ reads: α at the point x applied to the vectors $v_1, \ldots, v_p$.

<u>Note 1.</u> For $p = 0$ we have simply

$$\Omega^0(U) = \mathcal{C}^\infty(U, \mathbf{R}).$$

<u>Note 2.</u> The set $\Omega^p(U)$ is obviously a vector space. It can be equipped with a functional diffeology that extends the functional diffeology on $\Omega^0(U) = \mathcal{C}^\infty(U, \mathbf{R})$.

<u>**86. Pullback of a smooth form.**</u> Let $U \in \mathbf{R}^n$ and $U' \in \mathbf{R}^{n'}$, let $f \in \mathcal{C}^\infty(U, U')$ and $\beta \in \Omega^p(U')$. We call *pullback* de β by f, and we denote by $f^*(\beta)$ the p-form on U defined by:

$$f^*(\beta)(x)(v_1, \ldots, v_p) = \beta(f(x))(Mv_1, \ldots, Mv_p), \text{ with } M = D(f)(x),$$

for all $x \in U$ and all $v_1, \ldots, v_p \in \mathbf{R}^n$.

Then, $f^*(\beta)$ is a smooth p-form on U,

$$\forall \beta \in \Omega^p(U'), \quad f^*(\beta) \in \Omega^p(U).$$

Moreover f^* is a linear operator:

$$f^*(\lambda\alpha + \mu\beta) = \lambda f^*(\alpha) + \mu f^*(\beta),$$

and smooth for the functional diffeology.

87. Exterior derivative of a smooth form. Let α be a p-form,

$$\alpha \in \Omega^p(U),$$

on a domain $U \subset \mathbf{R}^n$.

Let $d\alpha$ be defined as follow:

$$\begin{aligned} d\alpha(x)(v_0, v_1, \ldots, v_p) &= \frac{\partial\alpha(x)(v_1, v_2, v_3, \ldots, v_p)}{\partial x}(v_0) \\ &- \frac{\partial\alpha(x)(v_0, v_2, v_3, \ldots, v_p)}{\partial x}(v_1) \\ &- \frac{\partial\alpha(x)(v_1, v_0, v_3 \ldots, v_p)}{\partial x}(v_2) \\ &- \frac{\partial\alpha(x)(v_1, v_2, v_0, \ldots, v_p)}{\partial x}(v_3) \\ &- \cdots \\ &- \frac{\partial\alpha(x)(v_1, v_2, \ldots, v_{p-1}, v_0)}{\partial x}(v_p). \end{aligned}$$

where $v_0, v_1, \ldots, v_p \in \mathbf{R}^n$.

Then,

$$d\alpha \in \Omega^{p+1}(U) \quad \text{and} \quad d[d\alpha] = 0,$$

for all α.

The $(p+1)$-form $d\alpha$ is called the *exterior derivative*, or *exterior differential*, of α, and the operator

$$d\colon \Omega^p(U) \to \Omega^{p+1}(U)$$

is called the *exterior differentiation*.

Note 1. for $f \in \Omega^0(U) = \mathcal{C}^\infty(U, \mathbf{R})$,

$$df(x)\colon v \mapsto \frac{\partial f(x)}{\partial x}(v).$$

Note 2. For $n = 2$ and $p = 1$

$$\alpha = a(x, y)dx + b(x, y)dy,$$

where $dx(v) = v^x$ and $dy(v) = v^y$ are the coordinate 1-forms, with $v = (v^x, v^y)$. We have:

$$d\alpha(x)(v_1, v_2) = \left(\frac{\partial b(x,y)}{\partial x} - \frac{\partial a(x,y)}{\partial y}\right)(v_1^x v_2^y - v_2^x v_1^y).$$

The linear 2-form

$$(v_1, v_2) \mapsto v_1^x v_2^y - v_2^x v_1^y$$

is denoted by $dx \wedge dy$, such that

$$d\alpha(x) = \left(\frac{\partial b(x,y)}{\partial x} - \frac{\partial a(x,y)}{\partial y}\right) dx \wedge dy.$$

Note 3. Pullback and exterior differentiation commute:

$$f^*(d\alpha) = d[f^*(\alpha)].$$

23. Differential forms in diffeology

88. Differential forms. Let X be a diffeological space. A differential p-form on X is a mapping

$$\alpha \colon \mathrm{P} \mapsto \alpha(\mathrm{P}),$$

for all plots P in X such that:

(1) $\alpha(\mathrm{P}) \in \Omega^p(\mathrm{U})$, with $\mathrm{U} = \mathrm{dom}(\mathrm{P})$.
(2) For all Euclidean domain V, for all $\mathrm{F} \in \mathcal{C}^\infty(\mathrm{V}, \mathrm{U})$,

$$\alpha(\mathrm{P} \circ \mathrm{F}) = \mathrm{F}^*(\alpha(\mathrm{P})).$$

The set of differential p-forms on X is vector space denoted by

$$\Omega^p(\mathrm{X}).$$

It can be equipped with a natural functional diffeology that extends the functional diffeology on

$$\Omega^0(\mathrm{X}) = \mathcal{C}^\infty(\mathrm{X}, \mathbf{R}).$$

Note. When we consider the Euclidean domain U as a diffeological space, then a differential form α is naturally identified with the value

$$\underline{\alpha} = \alpha(1_\mathrm{U}).$$

89. **Pulback of a differential form.** Let X and X' be two diffeological spaces. let $\beta \in \Omega^p(X')$ and $f \in \mathcal{C}^\infty(X, X')$. Then, for all plots P in X

$$[f^*(\beta)](P) = \beta(f \circ P)$$

defines the pullback of β by f,

$$\forall \beta \in \Omega^p(X'), \quad f^*(\beta) \in \Omega^p(X).$$

The pullback is linear and smooth.

Note. We can identify

$$\alpha(P) = P^*(\alpha).$$

We should more properly write $\alpha(P) = \underline{P^*(\alpha)} = P^*(\alpha)(1_U)$, where $U = \mathrm{dom}(P)$.

90. **Exterior differential of a differential form.** Let X be a diffeological space and $\alpha \in \Omega^p(X)$. Then,

$$d\alpha \colon P \mapsto d[\alpha(P)]$$

is a differential $(p+1)$-form on X. The operator d is linear and smooth, for the functional diffeology. It satisfies

$$d \circ d = 0.$$

24. Pushing forwards differential forms

91. **Pushing form onto quotients.** Let X and X' be two diffeological spaces. Let $\pi : X \to X'$ be a subduction and let α be a differential k-form on X.

Theorem. *The k-form α is the pullback of a k-form β defined on X', $\alpha = \pi^*(\beta)$, if and only if, for any two plots P and Q of X such that $\pi \circ P = \pi \circ Q$, then $\alpha(P) = \alpha(Q)$.*

We also say that β is the *pushforward* of α by π. The differential forms of X satisfying this property may be called *basic forms*, with respect to π.

Note 1. For every integer k, the pullback $\pi^* : \Omega^k(X') \to \Omega^k(X)$ is always a smooth linear map. The previous proposition is a characterization of the image of π^*, when π is a subduction.

Note 2. This property can be expressed with the help of a diagram and it is used for integrating closed 2-forms. Consider the pullback of π by itself

$$\pi^*(X) = \{(x_1, x_2) \in X \times X \mid \pi(x_1) = \pi(x_2)\},$$

with projections pr_1 and pr_2.

$$\begin{array}{ccc} \pi^*(X) & \xrightarrow{\mathrm{pr}_1} & X \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[\pi]{} & X' \end{array}$$

Then, α is basic with respect to π if and only if $\mathrm{pr}_1^*(\alpha) - \mathrm{pr}_2^*(\alpha) = 0$, in other words, if and only if $\mathrm{pr}_1^*(\alpha) = \mathrm{pr}_2^*(\alpha)$.

92. Vanishing forms on quotients. Let X and X′ be two diffeological spaces and $f\colon X \to X'$ be a subduction. Let α be a p-form on X′, $\alpha \in \Omega^p(X')$, $p \in \mathbf{N}$. Then, $f^*(\alpha) = 0$ if and only if $\alpha = 0$. Equivalently, for any two p-forms α and β on X′ and for every subduction f from X to X′,

$$f^*(\alpha) = f^*(\beta) \quad \Rightarrow \quad \alpha = \beta. \qquad (\blacklozenge)$$

In other words, for every subduction $f\colon X \to X'$, the pullback by $f^*\colon \Omega^p(X') \to \Omega^p(X)$ is injective.

Note. This implies in particular that if a diffeological space X has a finite dimension $n \in \mathbf{N}$, then every $n + k$ differential form, with $k > 0$, is zero. Formally,

$$\dim(X) = n < \infty \ \text{ implies } \ \Omega^{n+k}(X) = \{0\}, \text{ for all } k > 0. \qquad (\heartsuit)$$

93. The corner orbifold. Consider the *corner orbifold*

$$\mathcal{Q} = \Delta_1^2 = [\mathbf{R}/\{\pm 1\}]^2 = \mathbf{R}^2/\{\pm 1\}^2.$$

Show that every 2-form on $\mathcal{Q}$ is proportional to the 2-form ω defined on $\mathcal{Q}$ by

$$\pi^*(\omega) = 4xy\, dx \wedge dy.$$

That is, for any other differential form $\omega' \in \Omega^2(\mathcal{Q})$, there exists a smooth function $\phi \in \mathcal{C}^\infty(\mathcal{Q}, \mathbf{R})$ such that $\omega' = f\omega$.

☞ Proof. Let ω' be a 2-form on $\mathcal{Q}$, and let $\tilde{\omega}'$ be its pullback by π,

$$\tilde{\omega}' = \pi^*(\omega').$$

Thus, there exists a smooth real function F such that

$$\tilde{\omega}' = \mathrm{F} \times dx \wedge dy,$$

where $dx \wedge dy$ is the canonical basis on $\Omega^2(\mathbf{R}^2)$. But since $\pi \circ (\varepsilon, \varepsilon') = \pi$, for all $(\varepsilon, \varepsilon') \in \{\pm 1\}^2$, we get

$$\varepsilon\varepsilon' \mathrm{F}(\varepsilon x, \varepsilon' y) = \mathrm{F}(x, y),$$

for all $(x, y) \in \mathbf{R}^2$ and all ε, ε' in $\{\pm 1\}$. Thus, $\mathrm{F}(-x, y) = -\mathrm{F}(x, y)$ and $\mathrm{F}(x, -y) = -\mathrm{F}(x, y)$. In particular, $\mathrm{F}(0, y) = 0$ and $\mathrm{F}(x, 0) = 0$. Therefore, since F is smooth, there exists $f \in \mathcal{C}^\infty(\mathbf{R}^2, \mathbf{R})$ such that $\mathrm{F}(x, y) = 4xyf(x, y)$, with $f(\varepsilon x, \varepsilon' y) = f(x, y)$. Therefore, $\tilde{\omega}' = f \times \tilde{\omega}$, with

$$\tilde{\omega} = 4xy \times dx \wedge dy,$$

that is,

$$\tilde{\omega} = d(x^2) \wedge d(y^2),$$

but $x \mapsto x^2$ and $y \mapsto y^2$ are invariant by $\{\pm 1\}^2$, so they are the pullback by π of some smooth real functions on $\mathcal{Q}$. Thus, $d(x^2)$ and $d(y^2)$ are the pullback of 1-forms on $\mathcal{Q}$, let us say

$$d(x^2) = \pi^*(ds) \quad \text{and} \quad d(y^2) = \pi^*(dt),$$

so, $\tilde{\omega} = \pi^*(\omega)$, where $\omega = ds \wedge dt$ is a well defined 2-form on $\mathcal{Q}$. Now, since $f(\varepsilon x, \varepsilon' y) = f(x, y)$ means just that f is the pullback of a smooth real function ϕ on $\mathcal{Q}$, it follows that any 2-form ω' on $\mathcal{Q}$ is proportional to ω, that is, $\omega' = \phi \times \omega$, with $\phi \in \mathcal{C}^\infty(\mathcal{Q}, \mathbf{R})$. ▶

25. De Rham cohomology

<u>94. The De Rham cohomology.</u> Let X be a diffeological space. The exterior derivative defined above satisfies the *coboundary condition*

$$d : \Omega^p(\mathrm{X}) \to \Omega^{p+1}(\mathrm{X}), \;\; p \geq 0 \quad \text{and} \quad d \circ d = 0.$$

As is usual in cohomology theories **[McL75]**, when we have a chain complex — here the chain complex of real vector spaces $\Omega^\star(X) = \left\{\Omega^p(X)\right\}_{p=0}^{\infty}$ with a coboundary operator d — the space of *p-cocycles* is defined as the kernel in $\Omega^p(X)$ of the operator d, and the space of *p-coboundary* is defined as the image, in $\Omega^p(X)$, of the operator d. They will be denoted by

$$\begin{cases} Z^p_{\rm dR}(X) &= \ker\left[d : \Omega^p(X) \to \Omega^{p+1}(X)\right], \\ B^p_{\rm dR}(X) &= d(\Omega^{p-1}(X)) \subset Z^p_{\rm dR}(X) \quad \text{with} \quad B^0_{\rm dR}(X) = \{0\}. \end{cases}$$

The *De Rham cohomology groups* of X are then defined as the quotients of the spaces of cocycles by the spaces of coboundaries, we denote them by

$$H^p_{\rm dR}(X) = Z^p_{\rm dR}(X)/B^p_{\rm dR}(X).$$

Since the operator d is linear, and since the space of differential p-forms $\Omega^p(X)$, equipped with the functional diffeology, is a diffeological vector space, the De Rham cohomology group $H^p_{\rm dR}(X)$, equipped with the quotient diffeology, is a diffeological vector space.

95. Homotopy invariance of the De Rham cohomology. Let $f : X \to X'$ be a smooth map. Since the exterior derivative commutes with the pullback, we have a morphism

$$f^\# : H_{\rm dR}(X') \to H_{\rm dR}(X)$$

with

$$f^\#(\text{class}(\alpha')) = \text{class}(f^*(\alpha')),$$

for all $\alpha' \in \Omega^*(X')$. We shall prove the following theorem in a later lecture.

Theorem. *If $s \mapsto f_s$ is an homotopy, that is, a smooth path in $\mathcal{C}^\infty(X, X')$, then*

$$f_0^\# = f_1^\#.$$

That is called the *homotopy invariance* of the De Rham cohomology.

Corollary If A is a *deformation retract* of X, then the De Rham cohomology of X coincides with the De Rham cohomology of A. Contractible spaces have a trivial cohomology.

26. Cubic homology and De Rham cohomology

96. Cubes and cubic chains in diffeological spaces. A *standard p-cube* is the subset $[0,1]^p$ of $\mathbf{R}^p$, and we denote it by I^p,

$$\mathrm{I}^p = [0,1]^p \subset \mathbf{R}^p.$$

Let X be a diffeological space. A *smooth p-cube* in X any smooth map from $\mathbf{R}^p$ to X. And $\mathrm{Cub}_p(\mathrm{X})$ denotes the set of all the smooth p-cubes in X,

$$\mathrm{Cub}_p(\mathrm{X}) = \mathcal{C}^\infty(\mathbf{R}^p, \mathrm{X}).$$

The set $\mathrm{Cub}_p(\mathrm{X})$ will be equipped with the functional diffeology.

Note that since 0-cubes are any maps from $\mathrm{I}^0 = \mathbf{R}^0 = \{0\}$ to X, then $\mathrm{Cub}_0(\mathrm{X})$ is naturally equivalent to X, thanks to the diffeomorphism $x \mapsto \mathbf{x} = [0 \mapsto x]$. Hence,

$$\mathrm{Cub}_0(\mathrm{X}) \simeq \mathrm{X}.$$

Then, we define the *smooth cubic p-chains* in X, with coefficients in $\mathbf{Z}$, as the free Abelian group generated by $\mathrm{Cub}_p(\mathrm{X})$, and we denote it by $\mathrm{C}_p(\mathrm{X})$. Thus, a (smooth) cubic p-chain c, in X, is any finite $\mathbf{Z}$-linear combination of p-cubes, that is,

$$c = \sum_\sigma n_\sigma \sigma, \text{ with } \sigma \in \mathrm{Cub}_p(\mathrm{X}), \text{ and } n_\sigma \in \mathbf{Z},$$

where the sum is performed over a finite set of p-cubes called the *support* of c, and denoted by

$$\mathrm{Supp}(c) = \{\sigma \in \mathrm{Cub}_p(\mathrm{X}) \mid n_\sigma \neq 0\}.$$

The group of cubic p-chains $\mathrm{C}_p(\mathrm{X})$ can be represented by

$$\mathrm{C}_p(\mathrm{X}) \simeq \{c \in \mathrm{Maps}(\mathrm{Cub}_p(\mathrm{X}), \mathbf{Z}) \mid \#\,\mathrm{Supp}(c) < \infty\}.$$

Note that in the writing $\sum_\sigma n_\sigma \sigma$ of the chain c, $n_\sigma = c(\sigma)$. Then, the sum of two cubic p-chains c and c', and the multiplication of a cubic p-chain c by an integer m, are defined as usual:

$$(c + c')(\sigma) = c(\sigma) + c'(\sigma) \quad \text{and} \quad (mc)(\sigma) = m \times c(\sigma).$$

Note 1. A cubic chain can also be regarded as any finite family

$$\{(n_i, \sigma_i)\}_{i\in \mathcal{I}}$$

and can be written $\sum_{i\in \mathcal{I}} n_i\sigma_i$, with the convention that if $\sigma_i = \sigma_j$, then

$$\sum_{i\in \mathcal{I}} n_i\sigma_i = \sum_{i'\in \mathcal{I}'} n_{i'}\sigma_{i'} + (n_i + n_j)\sigma_i,$$

where $\mathcal{I}' = \mathcal{I} - \{i, j\}$. Since the family is finite, the sum of the coeff-cients of a same cube is finite and both aspects are equivalent.

Note 2. With smooth homology or cohomology in mind, there is no contradiction in defining smooth p-cubes in X as smooth maps from $\mathbf{R}^p$ to X, as we do here, or as the maps from I^p to X which are the restrictions of smooth maps defined on an open neighborhood of I^p, as we could have also chosen to do. Indeed the following proposition addresses this issue.

(♦) Every p-plot of X defined on a small open neighborhood of I^p coincides, on I^p, with some global p-plot of X.

This is why, for sake of simplicity and without loss of generality, the smooth p-cubes in X have been defined as the global p-plots of X. But to focus our attention on $\mathrm{I}^p \in \mathbf{R}^p$ we have introduced a special name, p-cube instead of global p-plot, and a special notation $\mathrm{Cub}_p(\mathrm{X})$ for $\mathcal{C}^\infty(\mathbf{R}^p, \mathrm{X})$.

97. Boundary of cubes and chains. Let us introduce the following family of injections, for all $a \in \mathbf{R}$:

$$j_k(\mathrm{a})\colon \mathbf{R}^p \to \mathbf{R}^{p+1},\ k = 1, \ldots, p+1,$$

defined by

$$\begin{array}{lllll} k = 1 & j_1(\mathrm{a})\colon (t_1)\cdots(t_p) & \mapsto & (\mathrm{a})(t_1)\cdots(t_p), \\ 1 < k \leq p & j_k(\mathrm{a})\colon (t_1)\cdots(t_p) & \mapsto & (t_1)\cdots(t_{k-1})(\mathrm{a})(t_k)\cdots(t_p), \\ k = p+1 & j_{p+1}(\mathrm{a}) : (t_1)\cdots(t_p) & \mapsto & (t_1)\cdots(t_p)(\mathrm{a}). \end{array}$$

Given a p-tuple of numbers, $j_k(\mathrm{a})$ puts a at the place number k, preserving the numbers before and shifting the numbers after.

We define the boundary operator ∂, for $p \geq 1$,

$$\text{for all } \sigma \in \mathrm{Cub}_p(\mathrm{X}), \quad \partial\sigma = \sum_{k=1}^{p} (-1)^k [\sigma \circ j_k(0) - \sigma \circ j_k(1)]. \qquad (\blacklozenge)$$

The operator ∂ defined by ($\blacklozenge$) is naturally extended by linearity on all cubic p-chains, with $p \geq 1$. The boundary of the p-chain $c = \sum_\sigma n_\sigma \sigma$ is given by

$$\partial c = \sum_\sigma n_\sigma \sum_{k=1}^{p} (-1)^k [\sigma \circ j_k(0) - \sigma \circ j_k(1)]. \qquad (\heartsuit)$$

The operator ∂ defined by ($\heartsuit$) is a boundary operator, that is,

$$\partial \circ \partial = 0,$$

and we get the *chain-complex*

$$\cdots \xrightarrow{\partial} \mathrm{C}_p(\mathrm{X}) \xrightarrow{\partial} \mathrm{C}_{p-1}(\mathrm{X}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathrm{C}_0(\mathrm{X}) \xrightarrow{\partial} \{0\}. \qquad (\clubsuit)$$

98. Degenerate cubes and chains. Let p and q be two integers such that $0 \leq q < p$. A *reduction* from $\mathbf{R}^p$ to $\mathbf{R}^q$ iq any projection $\mathrm{pr} : \mathbf{R}^p \to \mathbf{R}^q$ such that

$$\mathrm{pr}(t_1, \ldots, t_p) = (t_{i_1}, \ldots, t_{i_q}),$$

where $\{i_1, \ldots, i_q\} \subset \{1, \ldots, p\}$ is a subset of indices, and $i_1 < \cdots < i_q$. For $q = 0$, there is only one reduction: the constant map $\hat{0} : (t_1, \ldots, t_p) \mapsto 0$. So, a reduction from $\mathbf{R}^p$ to $\mathbf{R}^q$ consists of just "forgeting" some, or all, of the components of $t = (t_1, \ldots, t_p) \in \mathbf{R}^p$.

Now, let X be a diffeological space. Let $p > 0$ be an integer, we say that a p-cube $\sigma \in \mathrm{Cub}_p(\mathrm{X})$ is *degenerate* if there exists an integer q such that $0 \leq q < p$, a reduction pr from $\mathbf{R}^p$ to $\mathbf{R}^q$ and a q-cube $\sigma' \in \mathrm{Cub}_q(\mathrm{X})$ such that

$$\sigma = \sigma' \circ \mathrm{pr}.$$

In other words, a p-cube is degenerate if it does not depend on some coordinates of $\mathbf{R}^p$.

Let us denote by $\mathrm{Cub}_p^\bullet(\mathrm{X})$ the set of degenerate p-cubes of X, and let us denote by $\mathrm{C}_p^\bullet(\mathrm{X})$ the free Abelian group generated by $\mathrm{Cub}_p^\bullet(\mathrm{X})$.

The elements of $C^{\bullet}_{p}(X)$ are called the *degenerate cubic p-chains* of X. For $p = 0$, we agree that

$$\mathrm{Cub}^{\bullet}_{0}(X) = \varnothing \quad \text{and} \quad C^{\bullet}_{0}(X) = \{0\}.$$

We define the *reduced group of cubic p-chains of* X, denoted by $\mathbf{C}_p(X)$, as the quotient of the group of cubic p-chains of X by its subgroup of degenerate cubic p-chains, that is,

$$\mathbf{C}_p(X) = C_p(X)/\, C^{\bullet}_{p}(X).$$

Note that

$$\mathbf{C}_0(X) = C_0(X)/\, C^{\bullet}_{0}(X) = C_0(X)/\{0\} = C_0(X).$$

Now, for any integer $p > 0$, the boundary of any degenerate p-cube is a degenerate cubic p-chain, that is,

$$\text{for all } \sigma \in \mathrm{Cub}^{\bullet}_{p}(X), \quad \partial\sigma \in C^{\bullet}_{p-1}(X).$$

Then, by linearity we get immediately that

$$\text{for all } c \in C^{\bullet}_{p}(X), \quad \partial c \in C^{\bullet}_{p-1}(X) \ \text{ or } \ \partial[C^{\bullet}_{p}(X)] \subset C^{\bullet}_{p-1}(X).$$

Thus, there exists an operator, denoted again by ∂, from $\mathbf{C}_p(X)$ to $\mathbf{C}_{p-1}(X)$, such that the following diagram commutes

$$\begin{array}{ccc} C_p(X) & \xrightarrow{\ \partial\ } & C_{p-1}(X) \\ \pi_p \big\downarrow & & \big\downarrow \pi_{p-1} \\ \mathbf{C}_p(X) & \xrightarrow[\ \partial\ ]{} & \mathbf{C}_{p-1}(X) \end{array}$$

where π_p is the natural projection from $C_p(X)$ onto its quotient $\mathbf{C}_p(X)$. Moreover, the operator $\partial : \mathbf{C}_p(X) \to \mathbf{C}_{p-1}(X)$ again satisfies the boundary property $\partial \circ \partial = 0$.

99. Cubic homology. Let X be a diffeological space. As usual in homology theory [**McL75**], when we have a chain complex, here $\mathbf{C}_\star(X)$ with $\star = 0, 1, \ldots \infty$, and the boundary operator

$$\partial : \mathbf{C}_\star(X) \to \mathbf{C}_{\star-1}(X) \quad \text{with} \quad \partial \circ \partial = 0,$$

the space of *p-cycles* is defined as the kernel in $\mathbf{C}_p(X)$ of the operator ∂, and the space of *p-boundary* as the image, in $\mathbf{C}_p(X)$, of the operator ∂ defined on $\mathbf{C}_{p+1}(X)$. These spaces will be denoted by

$$\left\{ \begin{array}{lcll} \mathbf{Z}_p(X) & = & \ker[\partial\colon \mathbf{C}_p(X) \to \mathbf{C}_{p-1}(X)] & \text{with } p \geq 1, \\ \mathbf{B}_p(X) & = & \partial(\mathbf{C}_{p+1}(X)) \subset \mathbf{Z}_p(X) \subset \mathbf{C}_p(X) & \text{with } p \geq 0. \end{array} \right.$$

Then, the homology groups are defined as the quotients of the spaces of cycles by the spaces of boundaries

$$\mathbf{H}_p(X) = \mathbf{Z}_p(X) / \mathbf{B}_p(X).$$

Let us recall that for $p = 0$, $\partial\colon \mathbf{C}_0(X) \to \{0\}$, thus $\mathbf{Z}_0(X) = \mathbf{C}_0(X)$, and in this case $\mathbf{H}_0(X) = \mathbf{C}_0(X)/\partial\,\mathbf{C}_1(X) = C_0(X)/\partial\,\mathbf{C}_1(X)$. We call this homology $\mathbf{H}_*(X)$, the *cubic homology*[1] of the space X.

Once we have a homology, we get a cohomology, with values in $\mathbf{R}$ for example, by a standard procedure [TB, § 6.63].

A (real) *cubic p-cochain of* X is a linear map

$$f\colon\ C_p(X) \to \mathbf{R}$$

such that

$$f\colon \sum_\sigma n_\sigma \sigma \mapsto \sum_\sigma n_\sigma f(\sigma) \quad \text{and} \quad f \restriction \mathrm{Cub}_p(X) \in \mathcal{C}^\infty(\mathrm{Cub}_p(X), \mathbf{R}),$$

The spaces of cubic p-cochains is denoted by $C^p(X)$, that is,

$$C^p(X) = \mathrm{Hom}^\infty(C_p(X), \mathbf{R}) \simeq \mathcal{C}^\infty(\mathrm{Cub}_p(X), \mathbf{R}).$$

Now, the boundary ∂ defined from $C_{p+1}(X)$ to $C_p(X)$ induces, by duality, a *coboundary operator d* such that

$$d : C^p(X) \to C^{p+1}(X),$$

with

$$df(c) = f(\partial c)$$

[1]In topology the cubic homology and the singular homology coincide [**HY61**, Ex. 8.1] For a natural singular homology in diffeology, these two homologies will coincide also.

for all $f \in \mathrm{C}^p(\mathrm{X})$, and all $c \in \mathrm{C}_{p+1}(\mathrm{X})$. Then, by transfer of property, the coboundary d satisfies

$$d \in \operatorname{Hom}(\mathrm{C}^p(\mathrm{X}), \mathrm{C}^{p+1}(\mathrm{X})) \quad \text{and} \quad d \circ d = 0.$$

The cohomology groups are defined by considering this operation applied to the reduced cubic chains, that is, by considering cochains defined on $\mathbf{C}_p(\mathrm{X})$, or which is equivalent, by considering the cubic cochains modulo *reduced cochains*, that is, the ones vanishing on reduced chains. That gives finaly the *cubic cohomology groups* :

$$\mathbf{H}^p(\mathrm{X}, \mathbf{R}) = \mathbf{Z}^p(\mathrm{X}, \mathbf{R}) / \mathbf{B}^p(\mathrm{X}, \mathbf{R}).$$

Note. It is not clear if cubic (or singular) homology will play, in diffeology, the crucial role it plays in the theory of manifolds. But since it is a traditional tool, and since it is still a *smooth invariant*, it was worth extending it to the general case.

100. Integration on Chains. Let σ be a p-cube in X. Let α be a differential p-form on X, we *integrate* α on σ by

$$\int_\sigma \alpha = \int_{1_p} \alpha(\sigma) = \int_{\mathrm{I}^p} \sigma^*(\alpha)(1_p),$$

but $\sigma^*(\alpha)(1_p)$ is a smooth p-form on $\mathbf{R}^p$, thus, there exists a smooth real function $f \in \mathcal{C}^\infty(\mathbf{R}^p, \mathbf{R})$ such that

$$\sigma^*(\alpha)(1_p) = f \ \mathrm{vol}_p = f(x_1, \ldots, x_p)\, dx_1 \wedge \cdots \wedge dx_p.$$

and therefore,

$$\int_{\mathrm{I}^p} \sigma^*(\alpha)(1_p) = \int_{\mathrm{I}^p} f \,\mathrm{vol}_p = \int_0^1 dx_1 \cdots \int_0^1 dx_p\, f(x_1, \ldots, x_p).$$

Definition. *The integration of a differential p-form α on p-chains defines a cochain, denoted here by $\bar\alpha$:*

$$\bar\alpha(\sigma) = \int_\sigma \alpha \quad \Rightarrow \quad \bar\alpha\Big(\sum_\sigma n_\sigma \sigma\Big) = \sum_\sigma n_\sigma \bar\alpha(\sigma) = \sum_\sigma n_\sigma \int_\sigma \alpha.$$

It turns out that

Theorem. *The differential d on the cochain α coincides with the exterior derivative. This is the so-called* Stoke's theorem *which is* *actually due to Sir William Thomson (1824–1907), known as Lord Kelvin:*

$$\bar{\alpha}(\partial\sigma) = \overline{d\alpha}(\sigma), \quad \textit{that is,} \quad \int_\sigma d\alpha = \int_{\partial\sigma} \alpha.$$

That construction induces a morphism, called *De Rham Morphism* in every degree p

$$h_p \colon \mathrm{H}^p_{\mathrm{dR}}(\mathrm{X}) \to \mathbf{H}^p(\mathrm{X}, \mathbf{R}).$$

One can check that

$$\mathrm{H}^0_{\mathrm{dR}}(\mathrm{X}) = \mathbf{H}^0(\mathrm{X}, \mathbf{R}) = \mathrm{Maps}(\pi_0(\mathrm{X}), \mathbf{R}).$$

One can show also that

$$h_1 \colon \mathrm{H}^1_{\mathrm{dR}}(\mathrm{X}) \to \mathbf{H}^1(\mathrm{X}, \mathbf{R})$$

is injective. However, h_1 is not surjective as shows the example of the torus T_α :

$$\mathrm{H}^1_{\mathrm{dR}}(\mathrm{T}_\alpha) = \mathbf{R} \quad \text{and} \quad \mathbf{H}^1(\mathrm{T}_\alpha, \mathbf{R}) = \mathbf{R}^2.$$

The cokernel of h_1 in interpreted by the introduction of the Čech cohomology, see [PIZ23]. It is the class of principal fiber bundles with group $(\mathbf{R}, +)$ over X, equipped with a flat connection. This obstruction is the first characteristic class specific to diffeology, since every principal bundle with contractible fibre over a manifold is trivial. It is even a geometric interpretation of the cokernel of the morphism from $\mathrm{H}^1_{\mathrm{dR}}(\mathrm{X})$ to $\mathbf{H}^1(\mathrm{X}, \mathbf{R})$ in general.

$$d\omega = 0 \qquad \& \qquad \omega \neq d\alpha$$

Diffeology Fiber Bundles

This lecture talks about the theory of diffeology fiber bundles, which deviates from the usual locally trivial bundles, where local triviality is replaced in diffeology by local triviality along the plots. We illustrate this definition with a few examples where some new situations happen only in diffeology.

The question of what is a fiber bundle in diffeology arises immediately with the case of irrational torus T_α. A direct computation of the first homotopy group shows that the projection $T^2 \to T_\alpha$ behaves like a fibration, with fiber $\mathbf{R}$, but without being locally trivial, since T_α inherits the coarse topology. Indeed, from the diffeology we found directly that

$$\pi_0(T_\alpha) = \{T_\alpha\} \quad \text{and} \quad \tilde{T}_\alpha = \mathbf{R} \quad \text{with} \quad \pi_1(T_\alpha) = \mathbf{Z} + \alpha\mathbf{Z} \subset \mathbf{R},$$

where $\tilde{T}_\alpha$ plays the role of *universal covering* of T_α.

So, it was necessary to revise the notion of fiber bundle from classical differential geometry, and adapt it to diffeology in order to include these news objects, specific to diffeology, but without losing the main properties of this theory. This is what I have done in my ScD dissertation in 1985 [**Igl85**].

The main properties we wanted to preserve were:

(1) The homotopy long sequence, that we shall see in the lecture on homotopy.

(2) Any quotient class: $G \to G/H$ is a diffeological fiber bundle, with fiber H, where G is a diffeological group and H any subgroup.

These two properties, if satisfied, will explain the direct calculation of the homotopy of the irrational torus we did in [DI83].

In this lecture we shall see two equivalent definitions of *diffeological fiber bundles*. The first is pedestrian and operative, involving *local triviality along the plots*. The second one involving groupoids is more in the spirit of diffeology.

27. Diffeological fiber bundles, the pedestrian approach

101. Category of projections. We call *projection* any smooth surjection $\pi\colon Y \to X$, with X and Y two diffeological spaces. The space Y is called the *total space* of the projection and X is called the *base*. We define a category {Projections} whose obects are the projections and:

Definition 1. *The* morphisms *from* $\pi\colon Y \to X$ *to* $\pi'\colon Y' \to X'$, *are the pair of smooth maps* (Φ, ϕ) *such that:*

$$\Phi \in \mathcal{C}^\infty(Y, Y') \quad \textit{and} \quad \phi \in \mathcal{C}^\infty(X, X'),$$

such that

$$\begin{array}{ccc} Y & \xrightarrow{\Phi} & Y' \\ \pi\downarrow & & \downarrow\pi' \\ X & \xrightarrow[\phi]{} & X' \end{array} \quad \textit{with} \quad \pi' \circ \Phi = \phi \circ \pi.$$

The preimage

$$\pi^{-1}(x) = \{y \in Y \mid \pi(y) = x\}$$

is called the *fiber* of the projection over x. I use also the word *bundle* as synonym of projection, which will be specified in some cases in fiber bundle later, when some other properties will be satisfied.

Definition 2. *An* isomorphism *from* π *to* π' *will be a pair* (Φ, ϕ) *of diffeomorphisms.*

102. Pullbacks of bundles. Let $\pi \colon Y \to X$ be a projection. Let $f \colon X' \to X$ be a smooth map. We call *the total space of the pullback* of π by f, or simply the *pullback*, the space denoted and defined by

$$f^*(Y) = \{(x', y) \in X' \times Y \mid f(x) = \pi(y)\}.$$

This set is equipped with the subset diffeology of the product $X' \times Y$.

$$\begin{array}{ccc} f^*(Y) & \xrightarrow{\mathrm{pr}_2} & Y \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow[f]{} & X \end{array}$$

The pullback of π by f is the first projection:

$$\mathrm{pr}_1 \colon f^*(Y) \to X' \quad \text{with} \quad \mathrm{pr}_1(x', y) = x'.$$

103. Trivial projections. Projections on factors are particular cases of projections. We will see often the first projection of direct products:

$$\mathrm{pr}_1 \colon X \times F \to X$$

Definition. *We say that a projection* $\pi \colon Y \to X$ *is* trivial with fiber F *if it is isomorphic to the first projection* $\mathrm{pr}_1 \colon X \times F \to X$.

It is equivalent to say that there exists an isomorphism with the first projection $\mathrm{pr}_1 \colon X \times F \to X$ with type $(\Phi, 1_X)$.

$$\begin{array}{ccc} X \times F & \xrightarrow{\Phi} & Y \\ & {\scriptstyle \mathrm{pr}_1}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array}$$

And that is the way we will use it in general.

Proof. Assume that (Φ, ϕ) is an isomorphism from $\mathrm{pr}_1 \colon X \times F \to X$ to $\pi \colon Y \to X$. Consider then the isomorphism $(\phi^{-1} \times 1_F, \phi^{-1})$ from $\mathrm{pr}_1 \colon X \times F \to X$ to istelf, we get:

$$\begin{array}{ccccc} X\times F & \xrightarrow{\phi^{-1}\times 1_F} & X\times F & \xrightarrow{\Phi} & Y \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow[\phi^{-1}]{} & X & \xrightarrow[\phi]{} & X \end{array}$$

which gives

$$\begin{array}{ccccc} (x,y) & \xrightarrow{\phi^{-1}\times 1_F} & (\phi^{-1}(x),y) & \xrightarrow{\Phi} & \Phi(\phi^{-1}(x),y) \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle\pi} \\ x & \xrightarrow[\phi^{-1}]{} & \phi^{-1}(x) & \xrightarrow[\phi]{} & \phi(\phi^{-1}(x)) = x \end{array}$$

And that is the triangle diagram above. ▶

104. Locally trivial projections. With trivial projection comes *locally trivial* projection. Let $\pi\colon Y \to X$ a smooth projection as it is defined above.

Definition. *We say that the projection* π *is* locally trivial *if there exists a diffeological space* F*, a D-open covering* $(U_i)_{i\in\mathcal{I}}$ *of* X *such that the restrictions*

$$\pi_i\colon Y_i \to U_i \quad \textit{with} \quad Y_i = \pi^{-1}(U_i) \quad \textit{and} \quad \pi_i = \pi \restriction Y_i,$$

are trivial with fiber F.

It is equivalent to say that:

(1) π is *locally trivial at the point* $x \in X$, if there exists a D-open neighborhood U of x such that: the restriction $\pi_U : Y_U \to U$ is trivial, with $Y_U = \pi^{-1}(U)$ and $\pi_U = \pi \restriction Y_U$.
(2) π is *locally trivial everywhere* with fiber F.

105. Diffeological fiber bundles. A smooth projection $\pi\colon Y \to X$, is a *diffeological fibration*, or a *diffeological fiber bundle*, if it is *locally trivial along the plots*, that is, if the pullback of π by any plot P of X is locally trivial with some given fiber F. The space F is the *fiber* of the fibration.

That means precisely the following:

For all plots $P: U \to X$, for all $r \in U$, there exists an open neighborhood V of r in U such that $pr_1 : (P \restriction V)^*(Y) \to V$ is trivial with fiber F, that is, isomorphic to $pr_1 : V \times F \to V$. Recall that

$$P^*(Y) = \{(r, y) \in U \times Y \mid P(r) = \pi(y)\}.$$

Note 1. A diffeological fibration $\pi: Y \to X$ is, in particular, a subduction and even a *local subduction*. That is, for every plot $P: U \to X$, for all $r \in U$ and for all $y \in Y_x = \pi^{-1}(x)$, with $x = P(r)$, there exists a plot Q of Y defined on some open neighborhood V of r lifting $P \restriction V$, that is, $P \restriction V = \pi \circ Q$, and such that $Q(r) = y$.

Note 2. There is a hierarchy in the various notions of fiber bundles:

(1) Trivial fiber bundles are locally trivial (with respect to the D-topology).
(2) Locally trivial fiber bundles are locally trivial along the plots. The converse is not true.

To be locally trivial along the plots does not mean that the fibration itself is locally trivial, as many examples will point it out. For example, the projection of T^2 on the irrational torus $T_\alpha = T^2/\mathcal{S}_\alpha$ is locally trivial along the plots, with fiber $\mathbf{R}$, but not trivial. This is a particular case of quotient G/H of diffeological groups by a subgroup.

Note 3. If the base of a diffeological fiber bundle is a manifold, then the fiber bundle is locally trivial. This comes immediately from the definition, consider the pullback by local charts. If moreover the fiber is a manifold, then the diffeological fiber bundle is a fiber bundle in the category of manifolds. This shows in particular that the classical notion of fiber bundle can also be defined directly in diffeological terms as a property of its associated groupoid, as we shall see later, but of course this definition leads to leave an instant the category of manifolds.

106. Quotient of groups by subgroups. Let G be a diffeological group and $H \subset G$ be a subgroup. Then, the projection class: $G \to G/H$ is a diffeological fibration, where H acts on G by left multiplication denoted by $L(h)(g) = hg$.

Note 1. We recall that a diffeological group is a group equipped with a diffeology such that the multiplication $(g, g') \mapsto gg'$ and the inversion $g \mapsto g^{-1}$ are smooth.

Note 2. In particular the projection class: $T^2 \to T_\alpha = T^2/\mathcal{S}_\alpha$ is a diffeological fibration.

☞ Proof. Let $P : U \to G/H$ be a plot. We have:

$$P^*(G) = \{(r, g) \in U \times G \mid \mathrm{class}(g) = P(r)\}.$$

Let $r_0 \in U$, there exists an open neighborhood V of r_0 and a smooth lifting of P, $Q : r \mapsto g_r$, over V such that $\mathrm{class}(g_r) = P(r)$. Let

$$\psi : V \times H \to (P \restriction V)^*(G) \quad \text{defined by} \quad \psi(r, h) = (r, hg_r).$$

The inverse is given by

$$\psi^{-1} : (r, g) \mapsto (r, gg_r^{-1}).$$

Since the multiplication and the inversion are smooth, ψ and ψ^{-1} are smooth and ψ is an isomorphism from $V \times H$ to $(P \restriction V)^*(G)$, for pr_1 both sides. Therefore class: $G \to G/H$ is a diffeological fibration. ▶

107. Principal fiber bundles. Let X be a diffeological space and $g \mapsto g_X$ be a smooth action of a diffeological group G on X, that is, a smooth homomorphism from G to Diff(X) equipped with the functional diffeology of diffeological group.

Let $\mathcal{E}$ be the *action map*,

$$\mathcal{E} : X \times G \to X \times X \quad \text{with} \quad \mathcal{E}(x, g) = (x, g_X(x)).$$

Proposition. *If $\mathcal{E}$ is an induction, then the projection* class *from* X *to its quotient* X/G *is a diffeological fibration, with the group* G *as fiber. We shall say that the action of* G *on* X *is* principal.

Definition. *If a projection* $\pi : X \to Q$ *is equivalent to* class : $X \to G/H$,

$$\begin{array}{ccc} & X & \\ {}^{\text{class}}\swarrow & & \searrow^{\pi} \\ X/G & \xrightarrow[\phi]{} & Q \end{array}$$

that is, if there exists a diffeomorphism $\phi : G/H \to Q$ *such that* $\pi = \phi \circ \text{class}$*, then we shall say that* π *is a* principal fibration*, or a* principal fiber bundle*, with structure group* G*.*

Note a) If the action $\mathcal{E}$ is inductive, then it is in particular injective, which implies that the action of G on X is free, which is indeed a necessary condition.

Note b) The quotients class: $G \to G/H$ are principal bundles with group H.

Note c) Let $\pi\colon X \to Q$ the G-principal fiber bundle, then the pullback of π by any plot $P\colon U \to Q$ is locally trivial, let say with $\Phi\colon V \times G \to (P \restriction V)^*(X)$

$$\begin{array}{ccc} V \times G & \xrightarrow{\Phi} & (P \restriction V)^*(X) \\ {}_{\text{pr}_1}\searrow & & \swarrow_{\text{pr}_1} \\ & V & \end{array}$$

and write

$$\Phi(r, g) = (r, \Phi_r(g)).$$

Then, the isomorphism Φ satisfies the equivariant property:

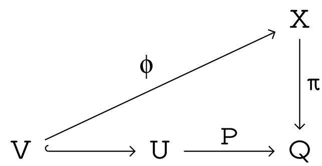
$$\Phi_r(gg') = g_X(\Phi_r(g')).$$

In particular,

$$\Phi_r(g) = g_X(\phi(r)) \quad \text{with} \quad \phi(r) = \Phi_r(1_G).$$

And ϕ is a local lifting of the plot P

$$\phi\colon V \to X \quad \text{and} \quad \pi \circ \phi = P \restriction V.$$



Note d) A principal fiber bundle is trivial if and only if it admits a global smooth section.

28. Diffeological fiber bundles, the groupoid approach

108. Structure groupoid of a projection. Let $\pi\colon Y \to X$ be a smooth surjection. Let us define

$$\mathrm{Obj}(K) = X \text{ and for all } x, x' \in X,\ \mathrm{Mor}_K(x, x') = \mathrm{Diff}(Y_x, Y_{x'}),$$

where the $Y_x = \pi^{-1}(x)$, $x \in X$, are equipped with the subset diffeology. Let us define on

$$\mathrm{Mor}(K) = \bigcup_{x,x' \in X} \mathrm{Mor}_K(x, x')$$

the natural multiplication $f \cdot g = g \circ f$, for $f \in \mathrm{Mor}_K(x, x')$ and $g \in \mathrm{Mor}_K(x', x'')$, K is clearly a groupoid. The source and target maps are given by

$$\mathrm{src}(f) = \pi(\mathrm{def}(f)) \quad \text{and} \quad \mathrm{trg}(f) = \pi(\mathrm{val}(f)).$$

The groupoid K is then equipped with a functional diffeology of K defined as follows. Let Y_{src} be the total space of the pullback of π by src, that is,

$$Y_{\mathrm{src}} = \{(f, x) \in \mathrm{Mor}(K) \times Y \mid x \in \mathrm{def}(f)\}.$$

We define the evaluation map ev as usual:

$$\mathrm{ev} : Y_{\mathrm{src}} \to Y \quad \text{with} \quad \mathrm{ev}(f, x) = f(x).$$

There exists a coarsest diffeology on Mor(K), which gives to K the structure of a diffeological groupoid and such that the evaluation map ev is smooth. It will be called again the *functional diffeology*. Equipped with this functional diffeology, the groupoid K captures

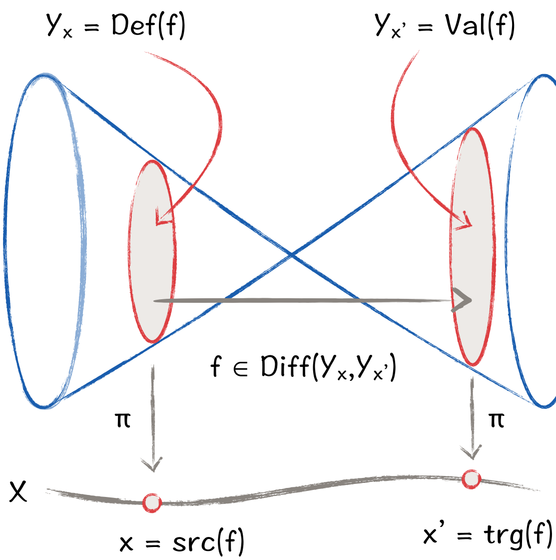


Figure 15. The groupoid associated with a projection.

the *smooth structure* of the projection π. It is why we define K as the *structure groupoid* of the surjection π.

Note 1. If X is reduced to a point, $\mathrm{Obj}(K) = \{\star\}$, this diffeology coincides with the usual functional diffeology of $\mathrm{Diff}(Y) = \mathrm{Mor}(K)$.

Note 2. This construction also applies when we have just a partition $\mathcal{P}$ on a diffeological space Y. We can equip the quotient $Q = Y/\mathcal{P}$ with the quotient diffeology, and we get the structure groupoid of the partition as the structure groupoid of the projection $\pi : Y \to Q$.

109. Fibrating groupoid. Let $\pi\colon Y \to X$ be a smooth projection.

The projection π is a *diffeological fibration* if the structure groupoid K is *fibrating*, that is, if and only if the characteristic map

$$\chi\colon \mathrm{Mor}(K) \to B \times B$$

is a subduction. In particular, the preimages $Y_x = \pi^{-1}(x)$, are necessarily all diffeomorphic since χ is surjective.

This definition is completely equivalent to the previous one § 105.

29. Associated fiber bundles

<u>**110. Associated fiber bundles.**</u> Consider a principal fiber bundle $\pi\colon Y \to X$ with group G. Let E be a diffeological space with a smooth action of G, that is a smooth morphism $g \mapsto g_E$ from G to Diff(E). Let the product over X be the quotient space

$$Y \times_G E = (Y \times E)/G,$$

where G acts on the product diagonally

$$g_{Y\times E}(y, e) = (g_Y(y), g_E(e)).$$

Then, the projection:

$$\mathrm{pr}\big(\mathrm{class}(y, e)\big) = \pi(y),$$

from $Y \times_G E$ to X is a diffeological fiber bundle with fiber E.

$$\begin{array}{ccc} Y \times E & \xrightarrow{\text{class}} & Y \times_G E \\ \mathrm{pr}_1 \downarrow & & \downarrow \mathrm{pr} \\ Y & \xrightarrow[\pi]{} & X \end{array}$$

<u>**111. Main theorem.**</u> Let $\pi\colon Y \to X$ be a diffeological fiber bundle with fiber E, then there exists a principal fiber bundle $\mathrm{pr}\colon T \to X$ with some diffeological group G acting smoothly on E, such that π is associate to pr. Actually, one can always take $G = \mathrm{Diff}(E)$.

30. Covering diffeological spaces

A special case of fiber bundle plays a special role in differential geometry, the covering spaces.

<u>**112. Covering spaces.**</u> Let X be a *connected* space, that is, for all pair of points $x, x' \in X$, there exists a *smooth path* $\gamma \in \mathrm{Paths}(X) = \mathcal{C}^\infty(\mathbf{R}, X)$ such that $\gamma(0) = x$ and $\gamma(1) = x'$. We say that x and x' are connected.

We call *covering* over X any fiber bundle such that the fiber is discrete (of course diffeologically). The covering may be non connected but we focus in general on connected coverings.

The main theorem about covering is about the universal covering.

113. Universal covering. Let X be a connected diffeological space, there exists a unique, up to equivalence, simply connected which is a principal fiber bundle whose group is the first group of homotopy $\pi_1(X)$. Any other connected covering is a quotient of this one.

Simple connexity will be defined in the next lecture.

114. Monodromy theorem. Let $f\colon Y \to X$ with X connected and Y simply connected. There exists always a smooth lifting $\tilde{f}\colon Y \to \tilde{X}$, where $\pi\colon \tilde{X} \to X$ is the universal covering. That is, $\pi \circ \tilde{f} = f$. And if we fix three points $y \in Y$, $x \in X$, $\tilde{x} \in \tilde{X}$, such that $f(y) = x$, $\pi(\tilde{x}) = x$, there exists a unique lifting such that $\tilde{f}(y) = \tilde{x}$.

115. Example: The irrational tori. If we call irrational tori any quotient $\mathcal{Q}$ of $\mathbf{R}^n$ by a discrete subgroup Γ that spans $\mathbf{R}^n$, that is, $\Gamma \otimes \mathbf{R} = \mathbf{R}^n$, then $\mathbf{R}^n$ is the universal covering and $\pi_1(\mathcal{Q}) = \Gamma$.

116. Example: Diff (S^1). The universal covering of group of diffeomorphisms $\mathrm{Diff}(S^1)$ is naturally identified as

$$\widetilde{\mathrm{Diff}}(S^1) \simeq \{\tilde{f} \in \mathrm{Diff}(\mathbf{R}) \mid \tilde{f}(x+1) = \tilde{f}(x) + 1\}$$

Any $\tilde{f} \in \widetilde{\mathrm{Diff}}(S^1)$ descends on $S^1 \simeq \mathbf{R}/2\pi\mathbf{Z}$ by

$$f(\mathrm{class}(x)) = \mathrm{class}(f(x)), \quad \text{or} \quad f(e^{2i\pi x}) = e^{2i\pi\tilde{f}(x)}.$$

31. Examples of diffeological fiber bundles

117. Example: The irrational flow on the torus. As said previously the projection class: $T^2 \to T_\alpha = T^2/\mathcal{S}_\alpha$ is a principal fibration with fiber $(\mathbf{R}, +)$. That is an interesting example because in usual differential geometry, the geometry of manifolds: every fiber bundle with a contractible fiber has a smooth global section. If it is a principal fiber bundle, it is then trivial. That is clearly not the case for diffeological fiber bundles since T^2 is not diffeomorphic to the product $T_\alpha \times \mathbf{R}$.

118. Example: The infinite sphere over the infinite projective space. We recall the construction of the infinite projective space, let

$$\mathbf{C}^\star = \mathbf{C} - \{0\} \quad \text{and} \quad \mathcal{H}^\star_{\mathbf{C}} = \mathcal{H}_{\mathbf{C}} - \{0\},$$

where $\mathcal{H}_{\mathbf{C}}$ is the Hilbert space of infinite square-summable sequences of complex numbers. We equip this space with the *fine diffeology* of vector space. The plots are the parametrizations that write locally

$$\mathrm{P}\colon r \mapsto \sum_{\alpha\in \mathrm{A}} \lambda_\alpha(r)\zeta_\alpha,$$

where A is a finite set of indices, the λ are smooth parametrizations in $\mathbf{C}$, and the ζ_α are fixed vector in $\mathcal{H}_{\mathbf{C}}$

Then, we consider the multiplicative action of the group $\mathbf{C}^\star$ on $\mathcal{H}^\star_{\mathbf{C}}$, defined by

$$(z, \mathrm{Z}) \mapsto z\mathrm{Z} \in \mathcal{H}^\star_{\mathbf{C}}, \quad \text{for all } (z, \mathrm{Z}) \in \mathbf{C}^\star \times \mathcal{H}^\star_{\mathbf{C}},$$

and the quotient of $\mathcal{H}^\star_{\mathbf{C}}$ by this action of $\mathbf{C}^\star$ is called the infinite complex projective space, denoted by

$$\mathcal{P}_{\mathbf{C}} = \mathcal{H}^\star_{\mathbf{C}}/\mathbf{C}^\star.$$

Now, $\mathcal{H}_{\mathbf{C}}$ is equipped with the fine diffeology and $\mathcal{H}^\star_{\mathbf{C}}$ with the subset diffeology. The infinite projective space $\mathcal{P}_{\mathbf{C}}$ is then equipped with the quotient diffeology. Let

$$\mathrm{class}\colon \mathcal{H}^\star_{\mathbf{C}} \to \mathcal{P}_{\mathbf{C}}$$

be the canonical projection. For every $k = 1, \ldots, \infty$, let

$$j_k : \mathcal{H}_{\mathbf{C}} \to \mathcal{H}^\star_{\mathbf{C}}$$

be the injections

$$j_1(\mathrm{Z}) = (1, \mathrm{Z}) \quad \text{and} \quad j_k(\mathrm{Z}) = (\mathrm{Z}_1, \ldots, \mathrm{Z}_{k-1}, 1, \mathrm{Z}_k, \ldots), \text{ for } k > 1.$$

Then, let F_k be the map

$$\mathrm{F}_k\colon \mathcal{H}_{\mathbf{C}} \to \mathcal{P}_{\mathbf{C}} \quad \text{with} \quad \mathrm{F}_k = \mathrm{class} \circ j_k, \quad k = 1, \ldots, \infty.$$

That is,

$$\mathrm{F}_1(\mathrm{Z}) = \mathrm{class}(1, \mathrm{Z})$$

and

$$F_k(Z) = \text{class}(Z_1, \ldots, Z_{k-1}, 1, Z_k, \ldots), \text{ for } k > 1.$$

Proposition. The preimage $\text{class}^{-1}(\text{val}(F_k)) \subset \mathcal{H}^\star_{\mathbf{C}}$ is isomorphic to the product $\mathcal{H}_{\mathbf{C}} \times \mathbf{C}^\star$, where the action of $\mathbf{C}^\star$ on $\mathcal{H}^\star_{\mathbf{C}}$ is transmuted into the trivial action on the factor $\mathcal{H}_{\mathbf{C}}$, and the multiplicative action on the factor $\mathbf{C}^\star$. Therefore, the projection

$$\text{class}: \mathcal{H}^\star_{\mathbf{C}} \to \mathcal{P}_{\mathbf{C}}$$

is a *locally trivial* $\mathbf{C}^\star$*-principal fibration*. We recall that a locally trivial fibration is stronger than a diffeological fiber bundle which is trivial only along the plots.

Proof. The subset $\text{class}^{-1}(\text{val}(F_k))$ is the set of $Z \in \mathcal{H}^\star$ equivalent to some $j_k(Z')$, with Z' any element in $\mathcal{H}$. That is, $Z = z.j_k(Z')$. Let then define

$$\Phi_k : \mathcal{H} \times \mathbf{C}^\star \to \mathcal{H}^\star \quad \text{by} \quad \Phi_k(Z, z) = z.j_k(Z).$$

This is a diffeomorphism from $\mathcal{H} \times \mathbf{C}^\star$ to $\text{class}^{-1}(\text{val}(F_k))$. On the other hand, we know that the $\text{val}(F_k)$, with $k \in \mathbf{N}$, is a D-open covering of $\mathcal{P}_{\mathbf{C}}$. Therefore, the projection $\text{class}: \mathcal{H}^\star_{\mathbf{C}} \to \mathcal{P}_{\mathbf{C}}$ is a principal bundle with groups $\mathbf{C}^\star$. ▶

119. Example: A remarkable non trivial fiber bundle. We know that, in the category of manifolds, a fiber bundle over a contractible manifold is trivial. It is also well known that in this category also, a fiber bundle with contractible fiber admits a smooth section. Therefore, that are two good reasons for a principal fiber bundle with a contractible group over a contractible manifold to be trivial.

The following example is an attempt to the construction of a diffeological fiber bundle over a contractible diffeological space, with a contractible group (that is $\mathbf{R}$), that would be not trivial.

We consider the action of $\mathbf{Z}$ on $\mathbf{C}$

$$\forall n \in \mathbf{Z}, \forall z \in \mathbf{C}, \quad \underline{n}(z) = ze^{2i\pi\alpha n},$$

with α an irrational number. Let

$$\mathcal{Q}_\alpha = \mathbf{C}/\underline{\mathbf{Z}}$$

Now, consider the following action of $\mathbf{Z}$ on $\mathbf{C} \times \mathbf{R}$

$$\underline{n}(z, t) = (\underline{n}(z), t + |z|^2 n)$$

Let

$$\mathbf{C} \times_{\underline{\mathbf{Z}}} \mathbf{R} = (\mathbf{C} \times \mathbf{Z})/\underline{\mathbf{Z}}.$$

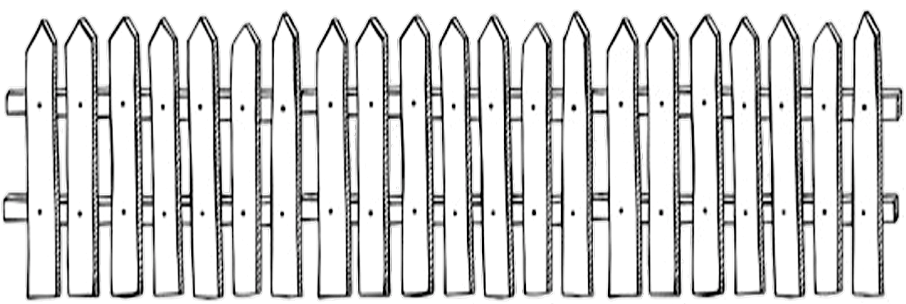
We have the following commutative diagram representing this construction:

$$\begin{array}{ccc} \mathbf{C} \times \mathbf{R} & \xrightarrow{\text{class}} & \mathbf{C} \times_{\underline{\mathbf{Z}}} \mathbf{R} \\ \Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow{\scriptstyle \pi} \\ \mathbf{C} \times \mathbf{R} & \xrightarrow[\text{class}]{} & \mathbf{C}/\underline{\mathbf{Z}} \end{array}$$

Then,

(1) The space $\mathbf{C}/\underline{\mathbf{Z}}$ is contractible.
(2) The fiber $\mathbf{R}$ of π is contractible.

Q. Is the projection $\pi \colon \mathbf{C} \times_{\underline{\mathbf{Z}}} \mathbf{R} \to \mathbf{C}/\underline{\mathbf{Z}}$ a $\mathbf{R}$-principal fiber bundle?



The proper use of fiber bundles...

Homotopy Theory in Diffeology

This lecture is about homotopy theory in diffeology, that generalizes to diffeological spaces the theory of homotopy on Euclidean domain, and encompasses the geometric theory of homotopy of manifolds. In these times when homotopy can have more than one meaning, this theory of homotopy should be understood as the historic version of homotopy, built from loops. We can specify if necessary and call it the "geometry homotopy theory of diffeological spaces".

We present the elementary constructions and definitions of the theory of homotopy in diffeology. In particular, the definitions of connectedness, connected components, homotopic invariants, the construction of the Poincaré groupoid, the fundamental group and the higher homotopy groups. We present the relative homotopy, and the exact sequence of the homotopy of a pair. Thanks to the functional diffeology on the space of paths of a diffeological space, we define the higher homotopy groups by considering simply the iteration of its space of loops. This theory has been originally presented in my doctoral dissertation *Fibrations difféologiques et homotopie* [**Igl85**].

32. Smooth paths and operations

120. Smooth paths in a diffeological space. We define the set of *smooth paths* in X, a diffeological space, as

$$\mathrm{Paths}(X) = \mathcal{C}^\infty(\mathbf{R}, X).$$

The origin and the end of the path γ are defined by

$$\hat{0}(\gamma) = \gamma(0) \quad \text{and} \quad \hat{1}(\gamma) = \gamma(1).$$

And the ends of γ are naturally defined by

$$\operatorname{ends}(\gamma) = (\gamma(0), \gamma(1)).$$

The set Paths(X) is equipped with the functional diffeology, then:

$$\hat{0}, \hat{1} \in \mathcal{C}^\infty(\mathrm{Paths}(X), X), \ \operatorname{ends} \in \mathcal{C}^\infty(\mathrm{Paths}(X), X \times X).$$

We generally say "path" for "smooth path", since we almost never consider not smooth paths.

Definition. *We say that x and x' are* connected *or* homotopic *if there exists a path such that:*

$$\operatorname{ends}(\gamma) = (x, x').$$

121. Smooth loops in a diffeological space. A *loop* in X is a path γ with same ends, that is, such that $\hat{0}(\gamma) = \hat{1}(\gamma)$. The space of loops is denoted by Loops(X),

$$\mathrm{Loops}(X) = \{\gamma \in \mathrm{Paths}(X) \mid \hat{0}(\gamma) = \hat{1}(\gamma)\}.$$

If we want to specify the *base point*,

$$\mathrm{Loops}(X, x) = \{\gamma \in \mathrm{Paths}(X) \mid x = \hat{0}(\gamma) = \hat{1}(\gamma)\}.$$

Unless otherwise stated, all subsets of Paths(X) are equipped with the subset diffeology.

122. Concatening paths. Let X be a diffeological space. We say that two paths γ and γ' are *juxtaposable* if

$$\hat{1}(\gamma) = \hat{0}(\gamma')$$

and if there exists a path $\gamma \vee \gamma'$ such that

$$\gamma \vee \gamma'(t) = \begin{cases} \gamma(2t) & \text{if} \quad t \leq \frac{1}{2}, \\ \gamma'(2t-1) & \text{if} \quad \frac{1}{2} \leq t. \end{cases}$$

The path $\gamma \vee \gamma'$ is called the *concatenation* of γ and γ'.

123. Reversing paths. Let X be a diffeological space. Let γ be a path in X, let $x = \hat{0}(\gamma)$ and $x' = \hat{1}(\gamma)$. The path

$$\bar{\gamma} : t \mapsto \gamma(1-t)$$

is called the *reverse path* of γ. It satisfies

$$\hat{0}(\bar{\gamma}) = \hat{1}(\gamma) \quad \text{and} \quad \hat{1}(\bar{\gamma}) = \hat{0}(\gamma).$$

The map

$$\text{rev} : \gamma \mapsto \bar{\gamma}$$

is smooth, it is an involution of Paths(X). If γ and γ' are juxtaposable, then $\text{rev}(\gamma')$ and $\text{rev}(\gamma)$ are juxtaposable, and

$$\text{rev}(\gamma') \vee \text{rev}(\gamma) = \text{rev}(\gamma \vee \gamma').$$

124. Stationary paths. Let X be a diffeological space. We say that a path γ is *stationary* at its ends, if there exists an open neighborhood of $]-\infty, 0]$ and an open neighborhood of $[+1, +\infty[$ where γ is constant. Formally, the path γ is stationary if there exists $\varepsilon > 0$ such that

$$\gamma \upharpoonright]-\infty, +\varepsilon[= [t \mapsto \gamma(0)] \quad \text{and} \quad \gamma \upharpoonright]1-\varepsilon, +\infty[= [t \mapsto \gamma(1)].$$

The set of stationary paths in X is denoted by

$$\text{stPaths(X)}.$$

The prefix st is used to denote everything stationary.

Note 1. Two stationary paths γ and γ' are juxtaposable if and only if $\hat{1}(\gamma) = \hat{0}(\gamma')$.

Note 2. The concatenation of stationary paths is not associative, if γ, γ' and γ'' are three stationary paths such that

$$\hat{1}(\gamma) = \hat{0}(\gamma') \quad \text{and} \quad \hat{1}(\gamma') = \hat{0}(\gamma''),$$

then $\gamma \vee (\gamma' \vee \gamma'')$ is *a priori* different from $(\gamma \vee \gamma') \vee \gamma''$. For a finite family of stationary paths $(\gamma_k)_{k=1}^{n}$ such that $\hat{1}(\gamma_k) = \hat{0}(\gamma_{k+1})$, with $1 \leq k < n$, we prefer, for reason of symmetry, the multiple concatenation defined by

$$\gamma_1 \vee \gamma_2 \vee \cdots \vee \gamma_n : t \mapsto \begin{cases} \gamma_1(nt - 1 + 1) & t \leq \frac{1}{n}, \\ \dots & \\ \gamma_k(nt - k + 1) & \frac{k-1}{n} \leq t \leq \frac{k}{n}, \\ \dots & \\ \gamma_n(nt - n + 1) & \frac{n-1}{n} \leq t, \end{cases}$$

which is still a stationary path, connecting $\hat{0}(\gamma_1)$ to $\hat{1}(\gamma_n)$.

<u>125. **Homotopic paths.**</u> Let X be a diffeological space. Because Paths(X) is itself a diffeological space, it makes sense to say that a path $s \mapsto \gamma_s$ in Paths(X) connects γ and γ', that is,

$$[s \mapsto \gamma_s] \in \mathrm{Paths}(\mathrm{Paths}(X)) = \mathcal{C}^\infty(\mathbf{R}, \mathrm{Paths}(X)),$$

with $\gamma_0 = \gamma$ and $\gamma_1 = \gamma'$.

• *Free-ends homotopy.* Such a path $\gamma \mapsto \gamma_s$ is called a free-ends homotopy, connecting γ to γ', or from γ to γ'.

• *Fixed-ends homotopy.* Now, let $\mathrm{Paths}(X, x, x')$ be the set of paths in X connecting x to x', equipped with the subset diffeology of Paths(X). A path $[s \mapsto \gamma_s] \in \mathrm{Paths}(\mathrm{Paths}(X, x, x')) = \mathcal{C}^\infty(\mathbf{R}, \mathrm{Paths}(X, x, x'))$ is called a *fixed-ends homotopy* from γ to γ'. But note that, by definition of the subset diffeology, $[s \mapsto \gamma_s]$ is a fixed-ends homotopy if and only if $[s \mapsto \gamma_s] \in \mathrm{Paths}(\mathrm{Paths}(X))$ and for all $s \in \mathbf{R}$, $\gamma_s(0) = x$ and $\gamma_s(1) = x'$.

<u>Proposition.</u> *A crucial property of homotopy in diffeology is that every path γ is fixed-ends homotopic to a stationary path.*

Indeed, let us consider the *smashing function* λ described by in Figure 16, where ε is some strictly positive real number, $0 < \varepsilon \ll 1$. The real function λ satisfies essentially the following conditions, and we can choose it increasing,

$$\lambda \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R}),\ \lambda \upharpoonright]-\infty, \varepsilon[= 0,\ \lambda \upharpoonright]1 - \varepsilon, +\infty[= 1.$$

Let $\gamma \in \mathrm{Paths}(X)$. We have the following properties:

a) The path $\gamma^\star = \gamma \circ \lambda$ is stationary with the same ends as γ.

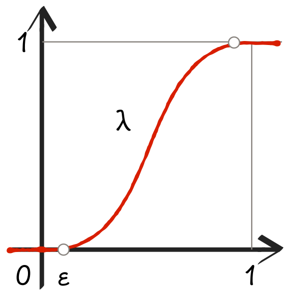


Figure 16. The smashing function λ.

b) The path γ is fixed-ends homotopic to $\gamma^\star$.

Note. As we know, not any two paths $\gamma, \gamma' \in \mathrm{Paths}(X)$ such that $\hat{1}(\gamma) = \hat{0}(\gamma')$ can be juxtaposable, but we can always force the concatenation by smashing them first. Hereafter, we shall use often this *smashed concatenation*, denoted and defined by

$$\gamma \star \gamma' = \gamma^\star \vee \gamma'^\star.$$

As a consequence of the point b),

Proposition. *If $\hat{1}(\gamma) = \hat{0}(\gamma')$, then $\gamma \star \gamma'$ connects $\hat{0}(\gamma)$ to $\hat{1}(\gamma')$. Moreover, if γ and γ' are juxtaposable, then $\gamma \vee \gamma'$ is homotopic to $\gamma \star \gamma'$.*

126. Connected components. Let X be a diffeological space.

Proposition 1. *To be connected, or homotopic, is an equivalence relation on X whose class are called* connected components. *The connected component of $x \in X$ is denoted by*

$$\mathrm{comp}(x) = \{x' \in X \mid \exists \gamma \in \mathrm{Paths}(X), \mathrm{ends}(\gamma) = (x, x')\}.$$

The set of components is denoted by

$$\pi_0(X) = \{\mathrm{comp}(x) \mid x \in X\}.$$

Equipped with the quotient diffeology, $\pi_0(X)$ is discrete.

Let $x \in X$, we denote by $\pi_0(X, x)$ the pointed space:

$$\pi_0(X, x) = (\pi_0(X), x).$$

Proposition 2. *If* X *is connected, then*

$$\text{ends}\colon \text{Paths}(X) \to X \times X$$

is a subduction.

127. The sum of its components. Let X be a diffeological space. The space X is the sum of its connected components. More precisely, if X is the sum of a family $\{X_i\}_{i\in \mathcal{I}}$, then the connected components of the X_i are the connected components of X. The decomposition of X into the sum of its connected components is the finest decomposition of X into a sum. It follows that the set of components $\pi_0(X)$, equipped with the quotient diffeology of X by the relation *connectedness*, is discrete.

128. Higher homotopy groups. Let X be a diffeological space, and let x be a point in X. The *higher homotopy groups* of X, based at x, are defined recursively. Let us denote by $\hat{x}$ the constant loop

$$\hat{x}\colon t \mapsto x.$$

Then, we define, for all integer $n > 0$:

$$\pi_n(X, x) = \pi_{n-1}(\text{Loops}(X, x), \hat{x}),$$

For $n = 1$ it gives

$$\pi_1(X, x) = \pi_0(\text{Loops}(X, x), \hat{x}) = (\pi_0(\text{Loops}(X, x)), \hat{x}),$$

which is called the *fundamental group* of X, based at x.

Proposition. *The set* $\pi_1(X, x)$ *is equipped with a multiplication defined by:*

$$\text{class}(\ell) \cdot \text{class}(\ell') = \text{class}(\ell \vee \ell'),$$

for all $\ell, \ell' \in \text{Loops}(X, x)$. *This multiplication gives* $\pi_1(X, x)$ *a structure of group:*

a) *the identity is* $\text{class}(\hat{x})$,
b) *the inverse of* $\text{class}(\ell)$ *is* $\text{class}(\bar{\ell})$, *where* $\bar{\ell}$ *is the reverse of* ℓ.

Proposition. *If* x *and* x' *are connected, then the groups* $\pi_1(X, x)$ *and* $\pi_1(X, x')$ *are conjugated.*

Now, let us define the recurrence

$$\begin{aligned} \mathrm{Loops}_{n+1}(X, x) &= \mathrm{Loops}(\mathrm{Loops}_n(X, x), \hat{x}_n), \\ \text{and } \hat{x}_{n+1} &= [t \mapsto \hat{x}_n], \end{aligned}$$

initialized by

$$\mathrm{Loops}_0(X, x) = X \text{ and } \hat{x}_0 = x.$$

We have then:

$$\pi_n(X, x) = \pi_0(\mathrm{Loops}_n(X, x), \hat{x}_n),$$

for all $n \in \mathbf{N}$. For $n \geq 1$, $\pi_n(X, x) = \pi_1(\mathrm{Loops}_{n-1}(X, x), \hat{x}_{n-1})$, which shows that the higher homotopy groups of X are the fundamental groups of some loop spaces, and therefore deserve their name of "group".

Note 1. This is specific to diffeology since in traditional differential geometry the set of loops of a manifold is not a manifold, and talking about the π_1 of a set of loops has no funded meaning.

For example, $\pi_2(X, x)$ is the fundamental group of the connected component of the constant loop $\hat{x}$ in $\mathrm{Loops}(X, x)$ etc. Since loop spaces are H-spaces, the groups $\pi_n(X, x)$ are Abelian for $n \geq 2$.

Note 2. Let $f : X \to X'$ be a smooth map, then f induced a map from $\pi_n(X, x)$ to $\pi_n(X', x')$, with $x' = f(x)$, which is a group morphism for all $n > 0$ and a morphism of pointed space for $n = 0$.

129. The Poincaré groupoid and fundamental group. Let X be a diffeological space. Let Π be the equivalence relation on Paths(X) :

$$\gamma \, \Pi \, \gamma' \;\Leftrightarrow\; \left\{ \begin{array}{l} \text{there exists } x, x' \in X \text{ and } \xi \in \mathrm{Paths}(\mathrm{Paths}(X, x, x')) \\ \text{such that } \xi(0) = \gamma \text{ and } \xi(1) = \gamma'. \end{array} \right.$$

Said differently, γ and γ' belong to the same component of

$$\mathrm{Paths}(X, x, x'),$$

the subspace of paths such that $\mathrm{ends}(\gamma) = \mathrm{ends}(\gamma') = (x, x')$. We shall denote by $\Pi(X)$ the diffeological quotient $\mathrm{Paths}(X)/\Pi$, and by class the canonical projection:

$$\mathrm{class}\colon \mathrm{Paths}(X) \to \Pi(X) = \mathrm{Paths}(X)/\Pi.$$

We shall denote again by ends the factorization of ends: $\mathrm{Paths}(X) \to X \times X$ on $\Pi(X)$.

Note that, if X is connected, then ends : $\Pi(X) \to X \times X$ is a subduction, and class $\upharpoonright$ $\mathrm{stPaths}(X) \to \Pi(X)$ is also a subduction.

The *Poincaré groupoid* $\mathbf{X}$ is then defined by

$$\mathrm{Obj}(\mathbf{X}) = X \quad \text{and} \quad \mathrm{Mor}(\mathbf{X}) = \Pi(X).$$

For all x and x' in X,

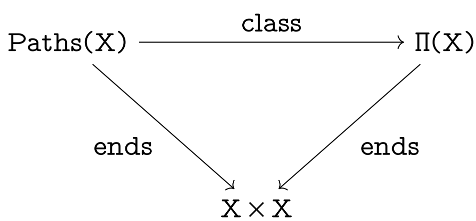
$$\mathrm{Mor}_{\mathbf{X}}(x, x') = \mathrm{Paths}(X, x, x')/\Pi = \pi_0(\mathrm{Paths}(X, x, x'))$$

is the set of fixed-ends homotopy classes of the paths connecting x to x'. The composition in the groupoid is the projection of the concatenation of paths. For all $\tau \in \mathrm{Mor}_{\mathbf{X}}(x, x')$ and $\tau' \in \mathrm{Mor}_{\mathbf{X}}(x', x'')$,

$$\tau \cdot \tau' = \mathrm{class}(\gamma \vee \gamma'), \quad \text{where} \quad \tau = \mathrm{class}(\gamma) \quad \text{and} \quad \tau' = \mathrm{class}(\gamma').$$

The paths γ and γ' are chosen in $\mathrm{stPaths}(X)$, for the concatenation $\gamma \vee \gamma'$ to be well defined.

The construction of the Poincaré groupoid is summarized by the following diagram.



The isotropy groups $\mathbf{X}_x = \mathrm{Mor}_{\mathbf{X}}(x, x)$ and the inverse of the elements of $\mathbf{X}(x, x') = \mathrm{Mor}_{\mathbf{X}}(x, x')$ are described by what follows:

a) For every point x of X, the isotropy group $\mathbf{1}_x \in \mathbf{X}_x$ is the component $\mathrm{class}(\hat{x})$, in $\mathrm{Loops}(X, x)$, of the constant path $\hat{x}$.

b) The inverse τ^{-1} of $\tau = \mathrm{class}(\gamma) \in \mathrm{Mor}_X(x, x')$ is the component $\mathrm{class}(\bar\gamma)$ of the reverse path $\bar\gamma = \mathrm{rev}(\gamma)$.

The structure group $\mathbf{X}_x = \mathrm{Mor}_X(x, x)$, where $x \in X$, is the first homotopy group, or the fundamental group, of X at the point x. That is, $\pi_1(X, x)$.

If X is connected, then the fundamental groups are isomorphic. They are precisely conjugate, if $\tau \in \mathrm{Mor}_X(x, x')$, then

$$\pi_1(X, x) = \tau \cdot \pi_1(X, x') \cdot \tau^{-1}.$$

In this case, the type of the homotopy groups $\pi_1(X, x)$ is denoted by $\pi_1(X)$.

Definition. *The space* X *is said to be* simply connected *if it is connected,* $\pi_0(X) = \{X\}$*, and if its fundamental group* $\pi_1(X)$ *is trivial. In that case* $\mathrm{ends}\colon \Pi(X) \to X \times X$ *is a diffeomorphism.*

Note. If X is not connected, then $\pi_1(X, x)$ is also the fundamental group of the connected component of x, that is, $\pi_1(X, x) = \pi_1(\mathrm{comp}(x), x) = \pi_1(\mathrm{comp}(x))$, since there is one type of fundamental group by component.

130. The universal covering. Let X be a connected diffeological space, let $\Pi(X) = \mathrm{Paths}(X)/\Pi$ the space of motphisms of the Poincaré groupoid.

Proposition. *The preimage*

$$\tilde X_x = \mathrm{ends}^{-1}(\{x\} \times X)$$

is a simply connected diffeological space. The projection

$$\pi = \hat 1 \restriction \tilde X_x$$

is a principal fiber bundle with group $\pi_1(X, x)$*. The spaces* $\tilde X_x$ *and* $\tilde X_{x'}$ *are equivalent and denoted generally by* $\tilde X$*.*

The space $\tilde X$ is called the *universal covering* of X.

131. Monodromy Theorem. Let $\pi\colon \tilde X \to X$ be the universal covering of a connected diffeological space. Let $f\colon Y \to X$ be a smooth map, where Y is simply connected.

Theorem. *There exists a smooth global lifting* $\tilde{f}: \mathrm{Y} \to \tilde{\mathrm{X}}$. *Let* $y \in \mathrm{Y}$, $x = f(y)$ *and* $\tilde{x} \in \pi^{-1}(x)$, *then there is a unique lifting* $\tilde{f}$ *such that* $\tilde{f}(y) = \tilde{x}$.

Proof. Let $y' \in \mathrm{Y}$ and $t \mapsto y_t$ a smooth path such that $y_0 = y$ and $y_1 = y'$. Let $x_t = f(y_t)$, $t \mapsto x_t$ is a path pointed at $x = f(y)$. Let

$$\tilde{f}(y') = \mathrm{class}[t \mapsto x_t].$$

This is a lift of f. And $\tilde{f}(y')$ does not depend on the special choice of the path $t \mapsto y_t$ because Y is simply connected. ▶

33. Every topological space admits a universal covering

For the topologist this sentence "Every topological space admits a universal covering" is incorrect, because every topologist is aware of the theorem:

Topology theorem *A pathwise connected topological space admits a simply connected covering if and only if it is semi-locally simply connected.*

However, as an application of the theory of diffeology homotopy and the previous construction of the Poincaré groupoid and subsequent universal covering, we get the following theorem, which is also true:

132. The universal covering of a topological space. Consider a pathwise connected topological space X, equip X with the *topo-diffeology* for which the plots are the continuous parametrizations. This diffeology was introduced at first by Paul Donato in his thesis [**Don84**].

Thus, as a diffeological space, X admits a simply connected diffeological covering $\pi: \tilde{\mathrm{X}} \to \mathrm{X}$, for which the projection π is continuous.

Indeed, the continuous paths, which are parametrizations by definition, are smooth for the topo-diffeology. Then, X equipped with the topo-diffeology is connected and the construction (§ 129) applies.

Therefore, every topological space admits a universal covering, in the sense of diffeology.

We have to note that the D-topology of the topo-diffeology is *a priori* finest than the initial topology. It contains *a priori* more open sets. Moreover, we shall see later that every diffeological space is locally path-connected, for the D-topology.

Note also that in diffeology the projection $\pi\colon \tilde{X} \to X$ is not *a priori* a local diffeomorphism, nor a local homeomorphism.

There are here many conjectures we can investigate about the relationship between this universal covering and the pure topological situation.

Diffeology gives us a procedure to smooth the space, even if it is not semi-locally simply conected, which is enough to give it a unique universal covering.

Let us recall that in topology a covering is a map $\pi\colon \tilde{X} \to X$ which is locally equivalent, in the sense of topology, to a direct product with a discrete fiber.

34. Relative homotopy

We describe now the homotopy of a pair (X, A), where X is a diffeological space and A is a subspace of X. We establish the short and long exact sequences of the homotopy of the pair (X, A), pointed at $a \in A$, which is a key ingredient of the exact homotopy sequence of the diffeological fiber bundles.

133. The short homotopy sequence of a pair. Let X be a diffeological space, let A be a subspace of X, and let $a \in A$. Let

$$\mathrm{Paths}(X, A, a) = \{\gamma \in \mathrm{Paths}(X) \mid \hat{0}(\gamma) \in A \text{ and } \hat{1}(\gamma) = a\}.$$

Let γ and γ' be two paths belonging to $\mathrm{Paths}(X, A, a)$, a *homotopy from* γ *to* γ'*, relative to* A*, pointed at* a is a path in $\mathrm{Paths}(X, A, a)$, connecting γ to γ'. We shall also call it an (A, a)-relative homotopy from γ to γ'. In Figure 17 the paths γ and γ' belong to $\mathrm{Paths}(X, A, a)$, with $A = A_1 \cup A_2$. The path γ is (A, a)-relatively homotopic to a loop in X, but not γ'. Let us consider the map

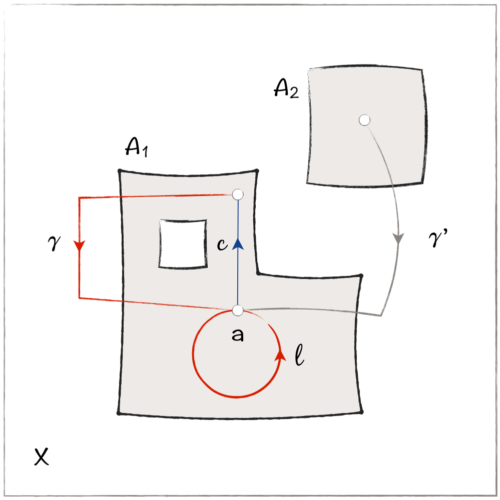


Figure 17. Relative homotopy of a pair.

$$\hat{0}\colon \text{Paths}(X, A, a) \to A$$

and the injection

$$i\colon A \to X.$$

They made up a two-terms sequence of smooth maps:

$$\text{Paths}(X, A, a) \xrightarrow{\hat{0}} A \xrightarrow{i} X.$$

This sequence induces naturally the two-terms sequence of morphisms of pointed spaces,

$$(\text{Paths}(X, A, a), \hat{a}) \xrightarrow{\hat{0}} (A, a) \xrightarrow{i} (X, a),$$

where $\hat{a} = [t \mapsto a]$. Then, this sequence induces a two-terms sequence on the space of components

$$\pi_0(\text{Paths}(X, A, a), \hat{a}) \xrightarrow{\hat{0}_\#} \pi_0(A, a) \xrightarrow{i_\#} \pi_0(X, a). \qquad (\heartsuit)$$

Note 1. $\ker(i_\#)$ — The *kernel* of $i_\#$ is the subset of components of A, contained in the component of X containing a.

Note 2. $\mathrm{val}(\hat{0}_\#)$ — The *values* of $\hat{0}_\#$ are the components of A, containing the initial points of the paths in X starting in A and ending at a. In other words, the subset of the components of A which can be connected, through X, to a. Now, it is clear that any component of A which can be connected to a by a path in X is included in the component of X containing a. Conversely, every component of A included in the component of X containing a can be connected to a by a path in X, starting in A. So, we get the equality,

$$\ker(i_\#) = \mathrm{val}(\hat{0}_\#).$$

Now, let us consider the inclusion of the triple (X, a, a) into (X, A, a). It induces an injection, denoted by j, on the space of paths,

$$\mathrm{Paths}(X, a, a) = \mathrm{Loops}(X, a) \xrightarrow{j} \mathrm{Paths}(X, A, a).$$

This injection descends, on the space of components, into a morphism of pointed spaces,

$$\pi_0(\mathrm{Loops}(X, a), \hat{a}) \xrightarrow{j_\#} \pi_0(\mathrm{Paths}(X, A, a), \hat{a}). \qquad (\blacklozenge)$$

Now, the concatenation of $(\blacklozenge)$ to the two-terms sequence $(\heartsuit)$ gives a three-terms sequence of morphisms of pointed spaces,

$$\begin{aligned}\pi_0(\mathrm{Loops}(X, a), \hat{a}) \xrightarrow{j_\#} \pi_0(\mathrm{Paths}(X, A, a), \hat{a})) \xrightarrow{\hat{0}_\#} \\ \pi_0(A, a) \xrightarrow{i_\#} \pi_0(X, a).\end{aligned} \qquad (\spadesuit)$$

Let us call, by abuse of language, *first group of homotopy of* X*, relative to* A*, pointed at* a, the pointed space denoted by $\pi_1(X, A, a)$, and defined by

$$\pi_1(X, A, a) = \pi_0(\mathrm{Paths}(X, A, a), \hat{a}).$$

Since, by definition, $\pi_0(\mathrm{Loops}(X, a), \hat{a}) = \pi_1(X, a)$ — regarded as pointed space — the sequence of morphisms $(\spadesuit)$ rewrites,

$$\pi_1(X, a) \xrightarrow{j_\#} \pi_1(X, A, a) \xrightarrow{\hat{0}_\#} \pi_0(A, a) \xrightarrow{i_\#} \pi_0(X, a). \qquad (\clubsuit)$$

This sequence is called the *short sequence of the relative homotopy of the pair* (X, A), *at the point* a. We have seen that

$$\ker(i_\#) = \mathrm{val}(\hat{0}_\#);$$

moreover,

$$\ker(\hat{0}_\#) = \mathrm{val}(j_\#).$$

Note 3. $\ker(\hat{0}_\#)$ — The kernel of $\hat{0}_\#$ is the set of the components of Paths(X, A, a) whose initial point belongs to the component of A containing a.

Note 4. $\mathrm{val}(j_\#)$ — The values of $j_\#$ are the components of the $\gamma \in$ Paths(X, A, a) which are (A, a)-relatively homotopic to some loops in X, based at a.

In short, the relative homotopy sequence of the pair (X, A), at the point a, is exact.

☞ Proof. We need only check that

$$\ker(\hat{0}_\#) = \mathrm{val}(j_\#).$$

Let us recall that, on the one hand, $\ker(\hat{0}_\#)$ is made up of the components of Paths(X, A, a) whose initial point belongs to the same component of A, containing a. On the other hand, $\mathrm{val}(j_\#)$ is the set of the components of paths $\gamma \in$ Paths(X, A, a) which are (A, a)-relatively homotopic to some loops in X, based at a.

Note 1. $\mathrm{val}(j_\#) \subset \ker(\hat{0}_\#)$. If a path γ is (A, a)-relatively homotopic to some loop in X based at a, its initial point is connected, in A, to a, and belongs to the same component of A containing a.

Note 2. $\ker(\hat{0}_\#) \subset \mathrm{val}(j_\#)$. Let us consider a component of A contained in the same component of X containing a. Let γ be a stationary path in X, beginning in A and ending at a such that its beginning belongs to the component of A containing a. Let $\gamma(0) = x$. Since x and a belong to the same component of A, there exists a stationary path c in A connecting a to x (Figure 17). Let $\xi(s) = [t \mapsto c(s+(1-s)\lambda(t))]$, where λ is the smashing function. Thus, ξ belongs to Paths(Paths(X, A, a)) and $\xi(s)(1) = c(1) = x = \gamma(0)$. So,

$\sigma(s) = \xi(s) \vee \gamma$ is a homotopy connecting $(c \circ \lambda) \vee \gamma \in \text{Loops}(X, a)$ to $\hat{x} \vee \gamma$, which is homotopic to γ. Therefore γ is (A, a)-relatively homotopic to a loop in X, based at a. ▶

134. The long homotopy sequence of a pair. Let X be a diffeological space, let A be a subspace of X, and let $a \in A$. Let us denote again by i the natural induction $i\colon \text{Loops}(A, a) \to \text{Loops}(X, a)$. There is no ambiguity with the injection i of § 133, since the spaces involved are not the same. Then, let us consider the two-terms sequence of smooth maps

$$\text{Paths}(\text{Loops}(X, a), \text{Loops}(A, a), \hat{a}) \xrightarrow{\hat{0}} \text{Loops}(A, a) \xrightarrow{i} \text{Loops}(X, a).$$

Or, if we prefer, by denoting

$$X_1 = \text{Loops}(X, a), \quad A_1 = \text{Loops}(A, a) \text{ and } a_1 = [t \mapsto a],$$

the above two-terms sequence of smooth maps writes

$$\text{Paths}(X_1, A_1, a_1) \xrightarrow{\hat{0}} A_1 \xrightarrow{i} X_1.$$

We can then apply the construction of the previous paragraph and get the short sequence of relative homotopy of the pair (X_1, A_1), at the point a_1. Let us denote again by j the natural induction from $\text{Loops}(X_1, a_1)$ to $\text{Paths}(X_1, A_1, a_1)$. Thus, we have,

$$\pi_1(X_1, a_1) \xrightarrow{j_\#} \pi_1(X_1, A_1, a) \xrightarrow{\hat{0}_\#} \pi_0(A_1, a_1) \xrightarrow{i_\#} \pi_0(X_1, a_1). \qquad (\blacklozenge)$$

Let us define the second group of relative homotopy of the pair (X, A), at the point a, by

$$\pi_2(X, A, a) = \pi_1(X_1, A_1, a_1) = \pi_0(\text{Paths}(X_1, A_1, a_1), a_1).$$

So, the short exact sequence $(\blacklozenge)$ writes now

$$\pi_2(X, a) \xrightarrow{j_\#} \pi_2(X, A, a) \xrightarrow{\hat{0}_\#} \pi_1(A, a) \xrightarrow{i_\#} \pi_1(X, a). \qquad (\heartsuit)$$

But the right term $\pi_1(X, a) = \pi_0(X_1, a_1)$ is just $\pi_0(\text{Loops}(X, a), \hat{a})$, that is, $\pi_1(X, a)$, regarded as a pointed space. It is also the left term of the relative homotopy sequence of the pair (X, A) at the point a. Let us connect the right term of the short homotopy sequences relative to the pair (X_1, A_1), to the left term of the short homotopy sequences relative to the pair (X, A). We get

$$\cdots \pi_2(X, A, a) \xrightarrow{\hat{\partial}_\#} \pi_1(A, a) \xrightarrow{i_\#} \pi_1(X, a) \xrightarrow{j_\#}$$
$$\pi_1(X, A, a) \xrightarrow{\hat{\partial}_\#} \pi_0(A, a) \cdots$$

Then, let us describe the connection of the morphisms of these two relative homotopy sequences at the junction $\pi_1(X, a)$.

Note 1. $\ker(j_\# : \pi_1(X, a) \to \pi_1(X, A, a))$ — This kernel is the set of classes of loops of X based at a which can be connected, relatively to (A, a), to the constant loop $\hat{a}$.

Note 2. $\mathrm{val}(i_\# : \pi_1(A, a) \to \pi_1(X, a))$ — This is the set of classes of loops in X, based at a, that are fixed-ends homotopic to a loop in A.

Now, if a loop in X, based at a, can be smoothly deformed into a loop contained in A, then it can be retracted relatively to A into the constant loop $\hat{a}$. Conversely, if a loop of X, based at a, is connected relatively to (A, a) to the constant loop $\hat{a}$, then it is fixed-ends homotopic to a loop in A. In other words,

$$\ker(j_\# \colon \pi_1(X, a) \to \pi_1(X, A, a)) = \mathrm{val}(i_\# : \pi_1(A, a) \to \pi_1(X, a)).$$

Thus, the connection of the two short exact relative homotopy sequences is exact. Now, let us define the *higher relative homotopy groups* of the pair (X, A) at the point a by recursion. Let us remark first that the inclusion $i \colon A \to X$ induces an inclusion

$$i_n : \mathrm{Loops}_n(A, a) \to \mathrm{Loops}_n(X, a).$$

Then, we can define

$$\mathrm{Paths}_{n+1}(X, A, a) = \mathrm{Paths}(\mathrm{Loops}_n(X, a), \mathrm{Loops}_n(A, a), \hat{a}_n),$$

for every integer n, and this gives the higher relative homotopy groups

$$\begin{aligned} \pi_{n+1}(X, A, a) &= \pi_0(\mathrm{Paths}_{n+1}(X, A, a), \hat{a}_n) \\ &= \pi_1(\mathrm{Loops}_n(X, a), \mathrm{Loops}_n(A, a), \hat{a}_n). \end{aligned}$$

Now, we can iterate the above connection of short relative homotopy sequences for each degree $n + 1 \to n$, and we get *the long exact relative homotopy sequence of the pair* (X, A)*, at the point* a.

$$\left.\begin{array}{l} \cdots \xrightarrow{i_\#} \pi_n(X,a) \xrightarrow{j_\#} \pi_n(X,A,a) \xrightarrow{\hat{0}_\#} \pi_{n-1}(A,a) \xrightarrow{i_\#} \pi_{n-1}(X,a) \cdots \\ \cdots \xrightarrow{i_\#} \pi_1(X,a) \xrightarrow{j_\#} \pi_1(X,A,a) \xrightarrow{\hat{0}_\#} \pi_0(A,a) \xrightarrow{i_\#} \pi_0(X,a). \end{array}\right\}$$

Proof. We need only check that

$$\ker(j_\# : \pi_1(X, a) \to \pi_1(X, A, a))$$

is equal to

$$\mathrm{val}(i_\# : \pi_1(A, a) \to \pi_1(X, a)).$$

Let us recall that $\ker(j_\#)$ is made up of the loops of X based at a which can be connected, relatively to (A, a), to the constant loop $\hat{a}$, and $\mathrm{val}(i_\#)$ is the subset of the classes of loops in X, based at a, which are fixed-ends homotopic to a loop in A.

<u>Note 1.</u> $\ker(j_\#) \subset \mathrm{val}(i_\#)$. Let ℓ be a loop in X, based at a, (A, a)-relatively homotopic to the constant loop $\hat{a}$. Let γ be the homotopy. So, for all $s \in \mathbf{R}$,

$$\gamma(0) = \ell, \quad \gamma(1) = \hat{a}, \quad \gamma(s)(0) \in A \ \text{ and } \ \gamma(s)(1) = a.$$

The properties of γ are schematized in Figure 18. Let us consider a line of $\mathbf{R}^2$, turning around the origin, its intersection with the cube describes a homotopy connecting ℓ to $\gamma_{t=0} \in \mathrm{Loops}(A, a)$. More precisely, let us consider first the path $\gamma' : s \mapsto [t \mapsto \gamma(t)(st)]$ in $\mathrm{Loops}(X, a)$. The path γ' connects ℓ to $[t \mapsto \gamma(t)(t)]$. Then, let us consider the path $\gamma'' : s \mapsto [t \mapsto \gamma((1-s)t)(t)]$ in $\mathrm{Loops}(X, a)$. The path γ'' connects $[t \mapsto \gamma(t)(t)]$ to $\gamma_{s=0} \in \mathrm{Loops}(A, a)$. Therefore, ℓ is (A, a)-relatively homotopic to a loop in A, based at a.

<u>Note 2.</u> $\mathrm{val}(i_\#) \subset \ker(j_\#)$. Let $\mathrm{comp}(\gamma) \in \mathrm{val}(i_\#)$. We can choose $\gamma \in \mathrm{Loops}(A, a)$. The path $\xi : s \mapsto [t \mapsto \gamma(s + (1-s)t)]$ is a (A, a)-relative homotopy connecting γ to the constant loop $\hat{a}$. ▶

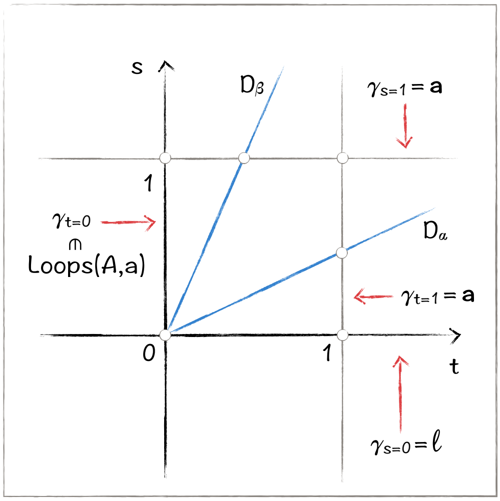


Figure 18. A relative homotopy of a loop to the constant loop.

135. The long homotopy sequence of a fiber bundle. Consider a diffeological fiber bundle $\pi\colon Y \to X$ with fiber F, then the long exact sequence of a pair induced a long exact sequence of the fiber bundle

$$\left.\begin{array}{l} \cdots \xrightarrow{i_\#} \pi_n(Y, y) \xrightarrow{\pi_\#} \pi_n(X, x) \xrightarrow{\partial} \pi_{n-1}(F, y) \xrightarrow{i_\#} \pi_{n-1}(Y, y) \cdots \\ \cdots \xrightarrow{j_\#} \pi_1(X, x) \xrightarrow{\partial} \pi_0(F, y) \xrightarrow{i_\#} \pi_0(Y, y) \xrightarrow{\pi_\#} \pi_0(X, x). \end{array}\right\}$$

Notes

Local Diffeology, Modeling

In this lecture we shall see how local diffeology builds a new branch of diffeology with the modeling process.

35. Local diffeology

136. Local smooth maps. We have seen what is a local smooth map

$$f : X \supset A \to X',$$

where X and X′ are two diffeological spaces. The map f is *local smooth* if for all plots $P : U \to X$, the following composite is a plot:

$$f \circ P : P^{-1}(A) \to X'$$

Proposition. *The composition of local smooth maps is a local smooth maps.*

Note that the composite of two local smooth map may be empty, the empty map is assumed to be smooth.

137. D-topology. We have seen that, if $f : X \supset A \to X'$ is a local smooth map, then for all plots $P \in \mathcal{D}$ (the diffeology of X) the preimage $P^{-1}(A)$ is open (an open subset of dom(P)). We then defined the *D-topology* as the finest topology on X such that the plots are continuous, that is,

- A subset $\mathcal{O} \subset X$ is *D-open* if $P^{-1}(\mathcal{O})$ is open for all plots in X.

Thus, the local smooth maps from X to X′ are the maps f defined on D-open subsets A of X such that restricted to A, $f \restriction A$ is smooth for the subset diffeology.

<u>138. Embedded subsets.</u> What is interesting with the D-topology, which is a perfect byproduct of the diffeology, is the definition of *embedding subsets* that result immediately, without the introduction of anything else.

Consider a subset $A \subset X$, and $j \colon A \to X$ be the inclusion. We have on A the subset diffeology of X, let us denote it by $\mathcal{D}_A = j^*(\mathcal{D})$, with $\mathcal{D}$ the diffeology of X.

We have also on X the D-topology T, and on A the D-topology T_A of $\mathcal{D}_A$.

But we have also the pullback $j^*(T)$ of the D-topology of X on A.

<u>Definition.</u> *We say that a subset* $A \subset X$ *is embedded in the diffeological space* X*, if the D-topology* T_A *of the subspace* A *coincides with the pullback* $j^*(T)$ *of the D-topology of* X *on* A.

$$T_A = j^*(T) \quad \Leftrightarrow \quad A \textit{ is embedded in } X.$$

In other words,

<u>Criterion</u> The subset $A \subset X$ is embedded if and only if for any D-open subset $\omega \subset A$, equipped with the subset diffeology, there exists an open subset $\Omega \subset X$, of the D-topology of X, such that $\omega = \Omega \cap A$.

<u>139. Example: The rational numbers.</u> The rational numbers $\mathbf{Q} \subset \mathbf{R}$ is discrete but not embedded. What is interesting here is that from a pure topological point of view, only embedded subgroups of $\mathbf{R}$ are regarded as discrete. They are all of the form $a\mathbf{Z}$, for any number a.

In diffeology it is more precise, we can have subgroups discrete and embedded, they coincide with the discrete subgroups from the topology point of view, and the discrete subgroup which are just induced but not embedded.

☞ Proof. We now that $\mathbf{Q}$ is discrete, that is, the plots are locally constant. Thus, any point $q \in \mathbf{Q}$ is open of the D-topology of the induced diffeology, since the pullback of q by a plot is a component of the domain of the plot, then open. I recall that to be locally constant for a plot means that to be constant on the connected componenents of the domain of the plot. Therefore,

Proposition. *The D-topology of a discrete diffeological space is discrete.*

Now, the intersection of an open subset of $\mathbf{R}$ with $\mathbf{Q}$ is always infinite, since $\mathbf{Q}$ is dense. Therefore, $\mathbf{Q}$ is not embedded. And we can conclude also that a strict subgroup $\Gamma \subset \mathbf{R}$, which is discrete, is embedded if and only if, for any element $\gamma \in \Gamma$ there is an interval $]\gamma-\varepsilon, \gamma+\varepsilon[$ such that $]\gamma-\varepsilon, \gamma+\varepsilon[\cap \Gamma = \{\gamma\}$. Hence there exists a smallest element $0 < a$ in Γ, and therefore $\Gamma = a\mathbf{Z}$. ▶

140. Embeddings. Let A and X be two diffeological spaces and $j\colon \mathrm{A} \to \mathrm{X}$ be a map. We say that j is an embedding if

(1) j is an induction.
(2) $j(\mathrm{A}) \subset \mathrm{X}$ is embedded.

141. Example: The group $\mathrm{GL}(n, \mathbf{R})$. Consider the group of linear isomorphisms $\mathrm{GL}(n, \mathbf{R}) \subset \mathrm{Diff}(\mathbf{R}^n)$. The group of diffeomorphisms is equipped with the functional diffeology of group of diffeomorphisms, that is, a parametrization $r \mapsto f_r$ in $\mathrm{Diff}(\mathbf{R}^n)$, defined on U, is smooth if and only if:

(1) $(r, x) \mapsto f_r(x)$, defined on $\mathrm{U} \times \mathbf{R}^n$ is a plot in $\mathbf{R}^n$.
(2) $(r, x) \mapsto (f_r)^{-1}(x)$, defined on $\mathrm{U} \times \mathbf{R}^n$ is a plot in $\mathbf{R}^n$.

As a subset of $\mathrm{Diff}(\mathbf{R}^n)$, $\mathrm{GL}(n, \mathbf{R})$ inherits the functional diffeology.

On the other hand, the group $\mathrm{GL}(n, \mathbf{R})$ is the open subset of $\mathbf{R}^{n\times n}$:

$$\mathrm{GL}(n, \mathbf{R}) = \{(m_{ij})_{i,j=1}^n \mid m_{ij} \in \mathbf{R} \text{ and } \det((m_{ij})_{i,j=1}^n) \neq 0\}.$$

Proposition. *The injection $j\colon \mathrm{GL}(n, \mathbf{R}) \to \mathrm{Diff}(\mathbf{R}^n)$ is an embedding.*

☞ Proof. First of all, j is injective.

Let us prove that j is an induction. Let $r \mapsto f_r$ be a plot in $\mathrm{Diff}(\mathbf{R}^n)$ with values in $\mathrm{GL}(n, \mathbf{R})$. Let $\boldsymbol{e}_i$ be the canonical basis of $\mathbf{R}^n$ and $\boldsymbol{e}_i^*$ the dual basis. Then, the coefficient of f_r are given by $m_{ij}(r) = \mathbf{e}_i^*(f_r(\mathbf{e}_j))$. They are obviously smooth, by definition of the functional diffeology.

Now, let us prove that j is an embedding. Consider the open ball $\mathrm{B}(\mathbf{1}_n, \varepsilon)$, centered at the identity and of radius ε. Let Ω_ε be the set of all diffeomorphisms defined by

$$\Omega_\varepsilon = \{f \in \mathrm{Diff}(\mathbf{R}^n) \mid \mathrm{D}(f)(0) \in \mathrm{B}(\mathbf{1}_n, \varepsilon)\},$$

where $\mathrm{D}(f)(0)$ is the tangent linear map of f at the point 0. Now, let us prove the following:

(a) The set Ω_ε is open for the D-topology of $\mathrm{Diff}(\mathbf{R}^n)$.

Let $\mathrm{P} : \mathrm{U} \to \mathrm{Diff}(\mathbf{R}^n)$ be a plot, that is, $[(r, x) \mapsto \mathrm{P}(r)(x)] \in \mathcal{C}^\infty(\mathrm{U} \times \mathbf{R}^n, \mathbf{R}^n)$. The pullback of Ω_ε by P is the set of $r \in \mathrm{U}$ such that the tangent map $\mathrm{D}(\mathrm{P}(r))(0)$ is in the ball $\mathrm{B}(\mathbf{1}_n, \varepsilon)$, formally,

$$\mathrm{P}^{-1}(\Omega_\varepsilon) = \{r \in \mathrm{U} \mid \mathrm{D}(\mathrm{P}(r))(0) \in \mathrm{B}(\mathbf{1}_n, \varepsilon)\}.$$

Considering P as a smooth map defined on $\mathrm{U} \times \mathbf{R}^n$, $\mathrm{D}(\mathrm{P}(r))(0)$ is the partial derivative of P, with respect to the second variable, computed at the point $x = 0$. The map $[r \mapsto \mathrm{D}(\mathrm{P}(r))(0)]$ is then continuous, by definition of smoothness. Hence, the pullback of Ω_ε by this map is open. Because the imprint of this open set on $\mathrm{GL}(n, \mathbf{R})$ is exactly the ball $\mathrm{B}(\mathbf{1}_n, \varepsilon)$, we deduce that any open ball of $\mathrm{GL}(n, \mathbf{R})$ centered at $\mathbf{1}_n$ is the imprint of a D-open set of $\mathrm{Diff}(\mathbf{R}^n)$.

(b) Every open of $\mathrm{GL}(n, \mathbf{R})$ is the imprint of a D-open set of $\mathrm{Diff}(\mathbf{R}^n)$.

By using the group operation on $\mathrm{GL}(n, \mathbf{R})$ and since any open set of $\mathrm{GL}(n, \mathbf{R})$ is a union of open balls, every open subset of $\mathrm{GL}(n, \mathbf{R})$ is the imprint of some D-open subset of $\mathrm{Diff}(\mathbf{R}^n)$. Therefore, $\mathrm{GL}(n, \mathbf{R})$ is embedded in $\mathrm{Diff}(\mathbf{R}^n)$. ▶

142. Functional diffeology on local smooth maps. Let X and X′ be two diffeological spaces. Let $\mathcal{C}^\infty_{\mathrm{loc}}(\mathrm{X}, \mathrm{X}')$ be the set of local smooth maps from X to X′. The evaluation map is defined on

$$\mathfrak{F} = \{(f, x) \mid f \in \mathcal{C}^\infty_{\mathrm{loc}}(\mathrm{X}, \mathrm{X}') \text{ and } x \in \mathrm{dom}(f)\}$$

The evaluation map is, as usual,

$$\mathrm{ev}\colon \mathcal{C}^\infty_{\mathrm{loc}}(X, X') \times X \supset \mathfrak{F} \to X' \quad \text{with} \quad \mathrm{ev}(f, x) = f(x).$$

<u>Proposition.</u> *There exists a coarsest diffeology on* $\mathcal{C}^\infty_{\mathrm{loc}}(X, X')$ *such that the evaluation map is local smooth.*

That is, $\mathfrak{F}$ is a D-open subset of $\mathcal{C}^\infty_{\mathrm{loc}}(X, X') \times X$, and the map ev is smooth with $\mathfrak{F}$ equipped with the subset diffeology.

A parametrization $r \mapsto f_r$ in $\mathcal{C}^\infty_{\mathrm{loc}}(X, X')$, defined on U, is a plot if and only if the map

$$\psi\colon (r, x) \mapsto f_r(x)$$

defined on

$$\begin{aligned} P^*(\mathfrak{F}) &= \{(r, f, x) \in U \times X \mid f = f_r \text{ and } x \in \mathrm{dom}(f)\} \\ &\simeq \{(r, x) \in U \times X \mid x \in \mathrm{dom}(f_r)\} \end{aligned}$$

with value in X', is local smooth.

Note that $(r, x) \mapsto f_r(x)$ is the composite $\mathrm{ev} \circ \phi$, where $\phi(r, x) = (f_r, x)$.

$$\begin{array}{ccccc} U \times X \supset P^*(\mathfrak{F}) & \xrightarrow{\ \phi\ } & \mathfrak{F} & \xrightarrow{\ \mathrm{ev}\ } & X \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_1} & & \\ U & \xrightarrow{\ P\ } & \mathcal{C}^\infty_{\mathrm{loc}}(X, X') & & \end{array}$$

<u>**143. Example: Functional diffeology on D-open sets.**</u> Let X be a diffeological space. We get a diffeology on the set of D-open subsets of X as follow: consider a family of D-open subsets defined on some Euclidean domain U:

$$r \mapsto \mathcal{O}(r),$$

we can decide that the family is a plot in the set of D-open subsets of X if the map

$$r \mapsto 1_{\mathcal{O}(r)}$$

is a plot in the space of local smooth map. That is, if the subset

$$\mathcal{U} = \{(r, x) \in U \times X \mid x \in \mathcal{O}(r)\}$$

is a D-open subset on $U \times X$.

For example, let $r \mapsto I_r$ be a parametrization of open intervals in $\mathbf{R}$. This family is smooth if for all r_0 and $x_0 \in \mathbf{R}$ such that $x_0 \in I_{r_0}$, there exists a small ball B centered at r_0 and $\varepsilon > 0$ such that for all $r \in B$, $]x_0 - \varepsilon, x_0 + \varepsilon[\subset I_r$.

36. Manifolds

We shall present now the classical definition of *manifolds*, and then the diffeology way.

144. Manifolds, the classic way. We summarize the basic definitions, according to Bourbaki [**Bou82**], but we make the inverse convention, made also by some other authors, to regard charts defined from real domains to a manifold M, rather than from subsets of M into real domains.

(♣) Let M be a nonempty set. A *chart* of M is a bijection F defined on an n-domain U to a subset of M. The dimension n is a part of the data. Let $F : U \to M$ and $F' : U' \to M$ be two charts of M. The charts F and F' are said to be *compatible* if and only if the following conditions are fulfilled:

a) The sets $F^{-1}(F'(U'))$ and $F'^{-1}(F(U))$ are open.
b) The two maps $F'^{-1} \circ F : F^{-1}(F'(U')) \to F'^{-1}(F(U))$ and $F^{-1} \circ F' : F'^{-1}(F(U)) \to F^{-1}(F'(U'))$, each one the inverse of the other, are either empty or smooth. They are called *transition maps*.

An *atlas* is a set of charts, compatible two-by-two, such that the union of the values is the whole M. Two atlases are said to be compatible if their union is still an atlas. This relation is an equivalence relation. A *structure of manifold* on M is the choice of an equivalence class of atlases or, which is equivalent, the choice of a *saturated atlas*. Once a structure of manifold is chosen for M, every compatible chart is called a *chart of the manifold*.

145. Manifolds, the diffeology way. Let X be a diffeological space, we say that X is a *n-manifold* if it is locally diffeomorphic to $\mathbf{R}^n$ at all points. Such local diffeomorphisms from $\mathbf{R}^n$ to X are called *charts*. A generating family of charts is called an *atlas*.

The Euclidean domains are the first examples of manifolds.

146. Remark Where is the difference? The difference between the two definition gives an advantage to diffeology. The difference comes from that in diffeology, the set X is a priori equipped with a diffeology, that is a smooth structure. Then, the point is to test if the diffeology gives the space a structure of manifold.

A contrario, with the classical approach, the smooth structure is defined a posteriori. A parametrization P in a manifold M is smooth if the composite by the inverse of the charts is a smooth parametrization of $\mathbf{R}^n$.

Note that the same set equipped with two different diffeologies may give two different structures of manifolds with different dimensions. For example $\mathbf{R}^2$ can be equipped with its standard diffeology that gives it a structure of a 2-manifold. It can also be equipped with the sum diffeology $X = \sum_{x \in \mathbf{R}} \mathbf{R}$ which gives it a structure of 1-manifold.

147. Why these definitions give the same category? As we say previously, given a n-manifold M defined by the classic way, smooth parametrizations P in M are parametrizations such that $F^{-1} \circ P$ are smooth parametrizations in $\mathbf{R}^n$. We consider the empty parametrization as admissible. It is not difficult then to check that the set of these smooth parametrizations define a diffeology for which the charts are local diffeomorphisms. Conversely, local diffeomorphism from $\mathbf{R}^n$ to a diffeological manifold X define on X a structure of manifold, the classic way. And these two operations are inverse one from each other.

148. We know already some examples. Consider the sphere $S^2 \subset \mathbf{R}^3$. Consider the tangent plane at $N = (0, 0, 1)$, identified with $\mathbf{R}^2$, made of points

$$X = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Consider the projection

$$\mathrm{F}\colon \mathrm{X} \mapsto m \quad \text{with} \quad m = \frac{1}{\sqrt{x^2+y^2+1}} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

The map F is clearly injective from $\mathbf{R}^2$ into S^2, and smooth since x^2+y^2+1 never vanishes. Its inverse is given by

$$\mathrm{F}^{-1}\colon m \mapsto \mathrm{X} \quad \text{with} \quad \mathrm{X} = \begin{pmatrix} x = x'/z' \\ y = y'/z' \\ z = 1 \end{pmatrix} \quad \text{and} \quad m = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}.$$

We see here that necessarily $z' \neq 0$ and then F^{-1} is smooth. So, we got a local diffeomorphism F, around the North Pole N. Then, we use the transitive action of $\mathrm{SO}(3,\mathbf{R})$ to get a local diffeomorphism at all points in S^2.

To this example we have already seen the various tori $\mathrm{T}^n = \mathbf{R}^n/\mathbf{Z}^n$.

149. Diffeological manifolds. In diffeology we extend the definition of manifolds. A *diffeological manifolds* is a diffeological space locally diffeomorphic to a diffeological vector space at all points.

A *diffeology of vector space* is a diffeology on a vector space for which the addition and the multiplication by a scalar are smooth.

150. Example: The infinite complex projective space. We have seen the construction of the infinite projective space in § 118, where

$$\mathcal{P}_{\mathbf{C}} = \mathcal{H}^\star_{\mathbf{C}}/\mathbf{C}^\star$$

is equipped with the fine diffeology. We have introduced the maps F_k, with $k = 1, 2, \ldots$

$$\mathrm{F}_k\colon \mathcal{H}_{\mathbf{C}} \to \mathcal{P}_{\mathbf{C}} \quad \text{with} \quad \mathrm{F}_k = \mathrm{class} \circ j_k, \quad k = 1, \ldots, \infty.$$

That is,

$$\mathrm{F}_1(\mathrm{Z}) = \mathrm{class}(1, \mathrm{Z}) \quad \text{and}$$
$$\mathrm{F}_k(\mathrm{Z}) = \mathrm{class}(\mathrm{Z}_1, \ldots, \mathrm{Z}_{k-1}, 1, \mathrm{Z}_k, \ldots), \ \text{for } k > 1.$$

Then:

(1) For every $k = 1, \ldots, \infty$, j_k is an induction from $\mathcal{H}_{\mathrm{C}}$ into $\mathcal{H}^{\star}_{\mathrm{C}}$.

(2) For every $k = 1, \ldots, \infty$, F_k is a local diffeomorphism from $\mathcal{H}_{\mathrm{C}}$ to $\mathcal{P}_{\mathrm{C}}$. Moreover, their values cover $\mathcal{P}_{\mathrm{C}}$,

$$\bigcup_{k=1}^{\infty} \mathrm{val}(\mathrm{F}_k) = \mathcal{P}_{\mathrm{C}}.$$

Proposition. The diffeological space $\mathcal{P}_{\mathrm{C}}$ is a diffeological manifold modeled on $\mathcal{H}_{\mathrm{C}}$, for which the family $\{\mathrm{F}_k\}_{k=1}^{\infty}$ is an atlas.

37. Manifolds with boundary

151. Half-spaces. We denote by

$$\mathbf{H}_n = \mathbf{R}^{n-1} \times [0, \infty[$$

the *standard half-space* of $\mathbf{R}^n$.

We denote by $x = (r, t)$ its points with $r \in \mathbf{R}^{n-1}$ and $t \in [0, +\infty[$.

We denote by $\partial\mathbf{H}_n$ its boundary $\mathbf{R}^{n-1} \times \{0\}$. The subset diffeology of $\mathbf{H}_n$, inherited from $\mathbf{R}^n$, is made of all the smooth parametrizations $\mathrm{P} : \mathrm{U} \to \mathbf{R}^n$ such that $\mathrm{P}_n(r) \geq 0$ for all $r \in \mathrm{U}$, $\mathrm{P}_n(r)$ being the n-th coordinate of $\mathrm{P}(r)$. The D-topology of $\mathbf{H}_n$ is the usual topology defined by its inclusion into $\mathbf{R}^n$.

152. Smooth real maps from half-spaces. This is a theorem: A map $f \colon \mathbf{H}_n \to \mathbf{R}^p$ is smooth for the subset diffeology of $\mathbf{H}_n$ if and only if there exists an ordinary smooth map F, defined on an open neighborhood of $\mathbf{H}_n$, such that $f = \mathrm{F} \restriction \mathbf{H}_n$. Actually, there exists such an F defined on the whole $\mathbf{R}^n$.

Note. As an immediate corollary, any map f defined on $\mathcal{C} \times [0, \varepsilon[$ to $\mathbf{R}^p$, where $\mathcal{C}$ is an open cube of $\partial\mathbf{H}_n$, centered at some point $(r, 0)$, smooth for the subset diffeology, is the restriction of a smooth map $\mathrm{F} : \mathcal{C} \times]-\varepsilon, +\varepsilon[\to \mathbf{R}^p$.

Proof. First of all, if f is the restriction of a smooth map $\mathrm{F} \colon \mathbf{R}^n \to \mathbf{R}^p$, it is obvious that for every smooth parametrization $\mathrm{P} : \mathrm{U} \to \mathbf{H}_n$, $f \circ \mathrm{P} = \mathrm{F} \circ \mathrm{P}$ is smooth. Conversely, let f_i be a coordinate of f. Let

$x = (r, t) \in \mathbf{R}^{n-1} \times \mathbf{R}$. If f_i is smooth for the subset diffeology, then $\phi_i \colon (r, t) \mapsto f_i(r, t^2)$, defined on $\mathbf{R}^n$, is smooth. Now, ϕ_i is even in the variable t, $\phi_i(r, t) = \phi_i(r, -t)$. Thus, according to Hassler Whitney [Whi43, Theorem 1 and final remark], see Figure 19 and Figure 20, there exists a smooth map $F_i : \mathbf{R}^n \to \mathbf{R}$ such that: $\phi_i(r, t) = F_i(r, t^2)$. Hence, $f_i(r, t) = F_i(r, t)$ for all $r \in \mathbf{R}^{n-1}$ and all $t \in [0, +\infty[$. ▶

DIFFERENTIABLE EVEN FUNCTIONS

By Hassler Whitney

An even function $f(x) = f(-x)$ (defined in a neighborhood of the origin) can be expressed as a function $g(x^2)$; $g(u)$ is determined for $u \geq 0$, but not for $u < 0$. We wish to show that g may be defined for $u < 0$ also, so that it has roughly half as many derivatives as f. A similar result for odd functions is given.

Theorem 1. ***An even function $f(x)$ may be written as $g(x^2)$. If f is analytic, of class C^∞ or of class C^{2s}, g may be made analytic, of class C^∞ or of class C^s, respectively.***

Figure 19. Whitney theorem 1.

153. Half-spaces local diffeomorphisms. A map $f \colon A \to H_n$, with $A \subset H_n$, is a local diffeomorphism for the subset diffeology of $\mathbf{R}^n$ if and only if

(1) A is open in H_n,
(2) f is injective,
(3) $f(A \cap \partial H_n) \subset \partial H_n$,
(4) and for all $x \in A$ there exists an open ball $B \subset \mathbf{R}^n$ centered at x, and a local diffeomorphism $F \colon B \to \mathbf{R}^n$ such that f and F coincide on $B \cap H_n$.

Note. This implies in particular, that there exists an open neighborhood $\mathcal{U}$ of A and an étale application $g \colon \mathcal{U} \to \mathbf{R}^n$ such that f and g coincide on A.

☞ Proof. See [TB, § 4.14]. ▶

154. Classical manifolds with boundary. Manifolds with boundary have been precisely defined for example in [GP74] or in [Lee06], and others. We use here Lee's definition except that, for our subject, the direction of charts have been reversed.

Definition. A *smooth n-manifold with boundary* is a topological space M, together with a family of local homeomorphisms F_i defined on some open sets U_i of the half-space $\mathbf{H}_n$ to M, such that the values of the F_i cover M and, for any two elements F_i and F_j of the family, the transition homeomorphism $F_i^{-1} \circ F_j$, defined on $F_i^{-1}(F_i(U_i) \cap F_j(U_j))$ to $F_j^{-1}(F_i(U_i) \cap F_j(U_j))$, is the restriction of some smooth map defined on an open neighborhood of $F_i^{-1}(F_i(U_i) \cap F_j(U_j))$. The boundary ∂M is the union of the $F_i(U_i \cap \partial \mathbf{H}_n)$. Such a family $\mathcal{F}$ of homeomorphisms is called an atlas of M, and its elements are called *charts*. There exists a *maximal atlas* $\mathcal{A}$ containing $\mathcal{F}$, made with all the local homeomorphisms from $\mathbf{H}_n$ to M, such that the transition homeomorphisms with every element of $\mathcal{F}$ satisfy the condition given just above. We say that $\mathcal{A}$ gives to M its *structure of manifold with boundary*.

155. Manifolds with boundary, the diffeology Way. Let X be a diffeological space. We say that X is a n-manifold with boundary if it is locally diffeomorphic to the half-space $\mathbf{H}_n$ at all points. Such local diffeomorphisms are called charts of X and a set of charts that covers X is called an atlas.

Proposition. This definition is completely equivalent to the classic way above.

Note 1. Here again we shall note that the main difference is that the set X is a priori equipped with a diffeology, and we just check if its diffeology is a diffeology of manifold with boundary.

Note 2. The diffeology of a classic manifold with boundary M is defined by parametrizations in M such that, the composite with the inverse of all charts is smooth. That definition creates an equivalence between the classic and the diffeology categories.

38. Manifolds with corners

156. Corners. We denote by

$$K_n = [0, \infty[^n$$

the *standard corner* of R^n. That is, the subset

$$K_n = \{(x_1, \ldots, x_n) \mid x_i \geq 0, \forall i = 1 \ldots n\}.$$

The corner K_n is equipped with the subset diffeology inherited from R^n, which coincides with the nth-power of $[0, \infty[$. The plots are just the smooth parametrizations in P in R^n such that, for all $i = 1 \ldots n$ $P_i(r) \geq 0$. The D-topology of K_n is the usual topology defined by its inclusion into R^n.

> *Remark.* **Since g is constructed in a definite fashion, the theorems hold for functions of several variables which are even in one of them. (The case that f is of class C^∞ offers no further difficulty.) The reference above to [2] is to take care of this case.**

Figure 20. Whitney Last Remark.

157. Local smooth maps on corners. A map $f : K_n \to R^k$ is smooth for the subset diffeology if and only if, it is the restriction of a smooth map defined on an open neighborhood of K^n.

What does that mean pecisely?

Let $f : K_n \to R$ be a map such that: for every smooth parametrization $P : U \to R^n$ taking its values in K_n, $f \circ P$ is smooth. Then, f is the restriction of a smooth map F defined on some open neighborhood of K_n.

What doest that say?

That says that an heuristic consisting to define a smooth map from the corner K_n to R, as the restriction of a smooth map defined on an open neighborhood of K_n, can be avoided by using the diffeology framework. Assuming f to be smooth for the subset diffeology do the work, and moreover conceptually.

☞ Proof. The proof is a recurence on the same theorem above for half-spaces, see [GIZ19]. ▶

158. Local diffeomorphisms of corners. A local diffeomorphism f from K_n into itself is the restriction of an étale map defined on some open neighborhood of its domain of definition.

159. Classic manifolds with corners. Let M be a paracompact Hausdorff topological space. A n-chart with corners for M is a pair (U, ϕ), where U is an open subset of K^n, and ϕ is a homeomorphism from U to an open subset of M. Two charts with corners (U, ϕ) and (V, ψ) are said to be smoothly compatible if the composite map $\psi^{-1} \circ \phi \colon \phi^{-1}(\psi(V)) \to \psi^{-1}(\phi(U))$ is a diffeomorphism, in the sense that it admits a smooth extension to an open set in $\mathbf{R}^n$. An n-atlas with corners for M is a pairwise compatible family of n-charts with corners covering M. A maximal atlas is an atlas which is not a proper subset of any other atlas. An n-manifold with corners is a paracompact Hausdorff topological space M equipped with a maximal n-atlas with corners.

160. Diffeology manifolds with corners. A diffeological space X is a n-manifold with corners if and only if it is locally diffeomorphic to K_n at all points.

Manifold in the corner…

Modeling: Manifolds, Orbifolds and Quasifolds

In this lecture we build the category of orbifolds, and also quasifolds, by modeling locally these spaces according to what they should look like; and manifolds, of course.

Orbifolds have been introduced by Ishiro Satake as *V-Manifolds* in 1956 and 1957 [**Sat56**] [**Sat57**]. They have been presented as smooth structures for describing quotient spaces by a finite group of transformations. We shall recall Satake construction first and show then how we can rethink these spaces as diffeological spaces.

We show in particular how diffeology solves, in a pure geometrical way, a problem unresolved by Satake and successors, about what are smooth maps between orbifolds, and build then the subcategory {Orbifolds} in the catagory {Diffeology}.

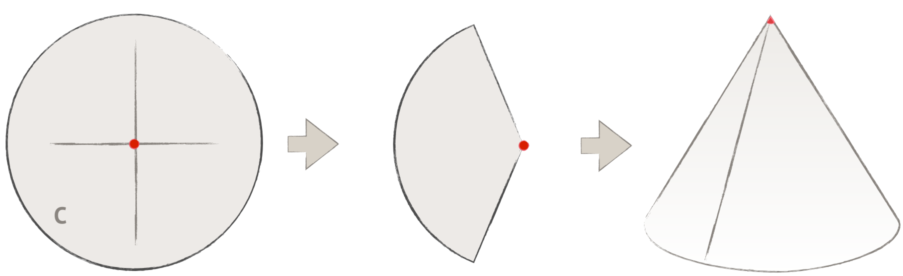


Figure 21. The cone orbifold viewed by a topologist.

Let us mention that the word *orbifold* has been coined by Thurston **[Thu78]** in 1978 as a substitue for *V-manifold*, but it recovers the same original Satake notion without modification. This concept was introduced to describe the smooth structure of spaces that look like manifolds, except on a few points or subsets, where they look like quotients of linear domains by a finite group of linear transformations.

The typical example is the quotient of the field **C** by a group of roots of unity.[1] The quotient space is always drawn as a cone as shown by Figure 21, to suggest the singularity of the point 0. But how do we capture the smooth structure around the singular point? That is the whole question.

39. Orbifolds, the Satake definition

The elementary brick in Satake construction is the Local Uniformization System. It is a topological construction.

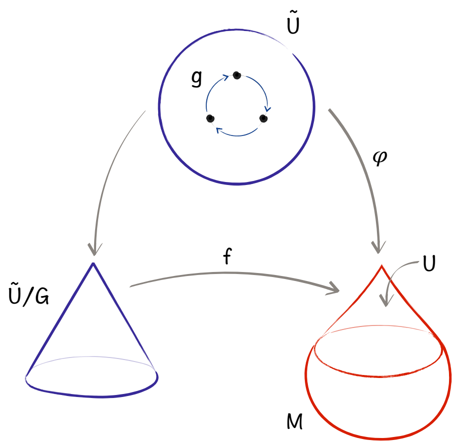


Figure 22. Local Uniformizing System.

[1]We consider **C** for its field structure: addition and multiplication, not for its complex structure which is anecdotic here.

161. Local uniformization system. Let M be a Hausdorff space and $U \subset M$ an open subset. A *local uniformizing system for* U (l.u.s) is a triple $(\tilde{U}, G, \varphi)$, where $\tilde{U}$ is a connected open subset of $\mathbf{R}^n$ for some n, where G is a finite group of diffeomorphisms of $\tilde{U}$,[2] and where $\varphi\colon \tilde{U} \to U$ is a map which induces a homeomorphism between $\tilde{U}/G$ and U.

Note. In Figure 22, f is this homeomorphism from $\tilde{U}/G$ to U.

Local uniformizing systems are patched together by *injections*; these can be thought of as the "transition maps". The following definition is taken from [**Sat57**, p. 466]:

162. Injections. An *injection* from an l.u.s $(\tilde{U}, G, \varphi)$ to another l.u.s $(\tilde{U}', G', \varphi')$ is a diffeomorphism λ from $\tilde{U}$ onto an open subset of $\tilde{U}'$ such that

$$\varphi = \varphi' \circ \lambda.$$

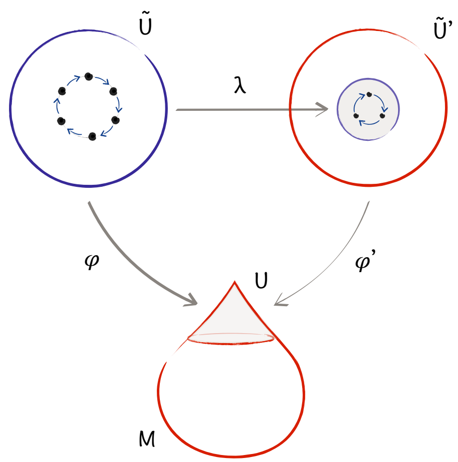


Figure 23. Injection.

163. Defining family. Let M be a Hausdorff space. A *defining family* on M is a family $\mathcal{F}$ of l.u.s for open subsets of M, satisfying conditions

[2] in original definition, the fixed point sets have codimension ≥ 2.

below. An open subset $U \subset M$ is said to be $\mathcal{F}$-*uniformized* if there exists an l.u.s. $(\tilde{U}, G, \varphi)$ in $\mathcal{F}$ such that $\varphi(\tilde{U}) = U$.

(1) Every point in M is contained in one $\mathcal{F}$-uniformized open set, at least. If a point p is contained in two $\mathcal{F}$-uniformized open sets U_1 and U_2, then there exists an $\mathcal{F}$-uniformized open set U_3 such that $p \in U_3 \subset U_1 \cap U_2$.
(2) If $(\tilde{U}, G, \varphi)$ and $(\tilde{U'}, G', \varphi')$ are l.u.s in $\mathcal{F}$ and $\varphi(\tilde{U}) \subset \varphi'(\tilde{U'})$, then there exists an injection $\lambda\colon \tilde{U} \to \tilde{U'}$.

In other words, if $p \in U \cap U'$ there exists a third l.u.s. $(\tilde{U''}, G'', \varphi'')$, two injections $\lambda\colon \tilde{U}'' \to \tilde{U}$ and $\lambda'\colon \tilde{U}'' \to \tilde{U}'$ such that $\phi' \circ \lambda' = \phi'' = \phi \circ \lambda$, as shown in Figure 24.

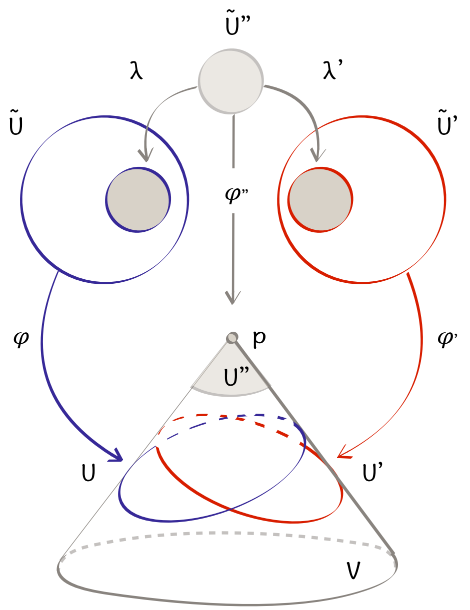


Figure 24. Defining Family.

164. V-manifold. The following definition is taken from [**Sat57**, p. 467, Definition 1].

Definition. *A* V-manifold *is a composite concept formed of a Hausdorff topological space* M *and a defining family* $\mathcal{F}$.

Two defining families $\mathcal{F}$ and $\mathcal{F}'$ are said to be *directly equivalent* if there exists a third defining family $\mathcal{F}''$ containing both of them. Two defining families are said to be *equivalent* if they are the ends of a chain of directly equivalent defining families. Equivalent families are regarded as defining one and the same V-manifold structure on M.

In [**Sat57**, p. 467, footnote 1] Satake write:

> *But in the following we consider a V-manifold* M *with a fixed defining family* $\mathcal{F}$. *(i.e. a* "coordinate V-manifold" $(M, \mathcal{F})$.

That is the convention we have made and when we say *V-manifold* it is always a coordinate V-manifold $(M, \mathcal{F})$ we have in mind.

40. Orbifolds as diffeologies

165. Diffeological orbifolds. Let X be a diffeological space. We say that X is an *diffeolgical n-orbifold* (or a D-orbifold) if X is everywhere locally diffeomorphic to some $\mathbf{R}^n/\Gamma$, with Γ a finite subgroup of $\mathrm{GL}(n, \mathbf{R})$, possibly different from point to point. The diffeological n-orbifolds are modeled on quotient spaces of type $\mathbf{R}^n/\Gamma$.

More precisely, for every point $x \in X$, there exists a finite group $\Gamma \subset \mathrm{GL}(n, \mathbf{R})$, a (connected) Γ-invariant Euclidean domain $\tilde{U} \subset \mathbf{R}^n$ and a local diffeomorphism $F : \tilde{U}/\Gamma \to X$ on a superset of x.

The situation, illustrated in Figure 25, looks like the previous definition of l.u.s except that here the quotient $\tilde{U}/\Gamma$ is equipped with the quotient diffeology, the map class: $\mathbf{R}^n \to \mathbf{R}^n/\Gamma$ is the canonical subduction and F is a local diffeomorphism.

These local diffeomorphisms are called *charts* of the D-orbifold X. An atlas of X is any covering set $\mathcal{A}$ of charts. Of course there exists a *saturated atlas* made of all charts.

Given an atlas $\mathcal{A}$ of X, that defines a generating family

$$\mathcal{F} = \{F \circ \text{class} \mid F \in \mathcal{A}\},$$

where class is relative to the group Γ associated with F. We call $\mathcal{F}$ the *strict generating family* associated with the atlas $\mathcal{A}$.

Note that when Γ is trivial we get the classical manifolds.

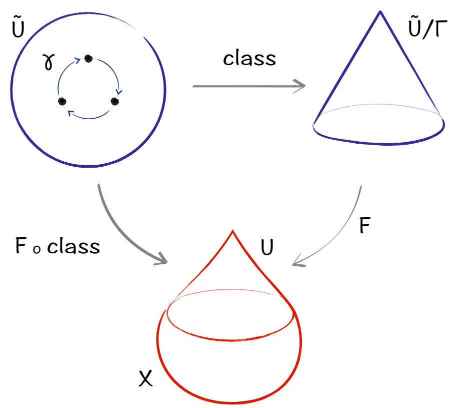


Figure 25. Generating Family for a D-orbifold.

166. Linear action or not? In the previous definition, it is equivalent to ask Γ to be a finite group of diffeomorphisms or to be linear. Indeed, we define a Γ-invariant Riemmannian metric by

$$\langle u, v\rangle_\Gamma = \frac{1}{\#\Gamma} \sum_{\gamma \in \Gamma} \langle \gamma u, \gamma v\rangle,$$

then the *slice theorem* states that this action is equivalent to an orthogonal action.

167. Example: The simplest. It is important to clarify a similar point we made for the diffeological definition of manifold. Here again an D-orbifold come a priori as a diffeological space, that is, equipped with a diffeology $\mathcal{D}$. The fact that X is an orbifold is a property of the diffeology, not the set itself, as the following examples will show.

The first example, the simplest, is certainly $\Delta_1 = \mathbf{R}/\{\pm 1\}$, which is equivalent to the half-line $[0, \infty[$ equipped with the pushforward of the standard diffeology of $\mathbf{R}$ by the map $x \mapsto x^2$. The D-topology is the subset topology on $[0, \infty[$. A plot of Δ_1 is a positive parametrization that can be writen locally everywhere as some $r \mapsto Q(r)^2$.

168. Example: The cone orbifold. Not the simplest example but the most known is the cone orbifold. It is defined as the quotient of the complex number space $\mathbf{C}$ by a cyclic group U_m. We denote it by:

$$\mathcal{C}_m = \mathbf{C}/U_m,$$

where $m \in \mathrm{N}$, $m \neq 0$ and

$$U_m = \{\exp(2i\pi k/m) \mid k = 1 \ldots m\}.$$

The diffeological space $\mathcal{C}_m$, equipped with the quotient diffeology, is by definition an orbifold. The topologists are used to represent this orbifold by gluing the two sides of a fundamental domain, as it is illustrated in Figure 21. But that representation disservices the diffeological intuition. We will show now how the orbifold $\mathcal{C}_m$ can be represented as a special diffeology on the field $\mathbf{C}$ itself.

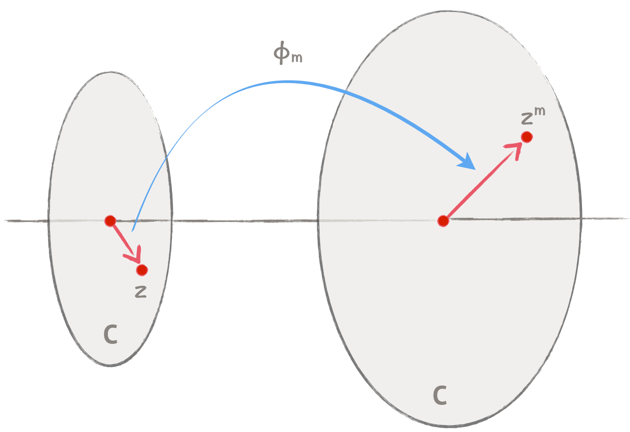


Figure 26. The cone orbifold viewed by a diffeologist.

Considering then the map

$$\phi_m : \mathbf{C} \to \mathbf{C} \quad \text{with} \quad \phi_m(z) = z^m,$$

it is clear that the preimages of the points $\zeta \in \mathbf{C}$ are exactly the orbits of the group U_m. Since ϕ_m is surjective, $\mathbf{C}$ can be identified with the quotient set $\mathbf{C}/U_m$, with canonical projection ϕ_m. The last question is what diffeology on $\mathbf{C}$ represents the quotient diffeology? Naturally, the answer is the pushforward

$$\mathcal{C}^\infty_m = \phi_{m*}(\mathcal{C}^\infty)$$

of the standard diffeology $\mathcal{C}^\infty$ on $\mathbf{C}$. Note that the D-topology of $\mathcal{C}^\infty_m$ is still the standard topology of $\mathbf{C}$.

A parametrization $P : U \to \mathbf{C}$ belongs to $\mathcal{C}^\infty_m$ if and only if, for all point $r_0 \in U$, there exists a small ball $\mathcal{B}$ centered at r_0 and a smooth parametrization $Q : \mathcal{B} \to \mathbf{C}$ such that $P(r) = Q(r)^m$, for all $r \in \mathcal{B}$.

Actually, if $P(r_0) \neq 0$, it is sufficient to ask P to be smooth on some small ball around r_0, and if $P(r_0) = 0$, then there is no shortcut, we have to find Q satisfying the condition above.

Thus, as we can see in this simple example, the same set $\mathbf{C}$ can be equipped with an inifinity of orbifold diffeologies, one for each integer, without altering the underlying space.

169. Example: The waterdrop. This orbifold, the waterdrop drawn in the many figures above, is a diffeology defined on the sphere S^2. By convenience, S^2 is regarded as a subset of $\mathbf{C} \times \mathbf{R}$. A plot of the *waterdrop diffeology* is a smooth parametrization ζ in S^2 which is identified to $\mathbf{C} \times \mathbf{R}$, with $N = (0, 1)$ the *North Pole*, satisfying:

$$\zeta : U \to \mathbf{C} \times \mathbf{R} \quad \text{with} \quad \begin{cases} \zeta(r) = \begin{pmatrix} z(r) \\ t(r) \end{pmatrix}, \\ |z(r)|^2 + t(r)^2 = 1. \end{cases}$$

such that, for all $r_0 \in U$:

- if $\zeta(r_0) \neq N$, then there exists a small ball $\mathcal{B}$ centered at r_0 such that $\zeta \restriction \mathcal{B}$ is smooth.

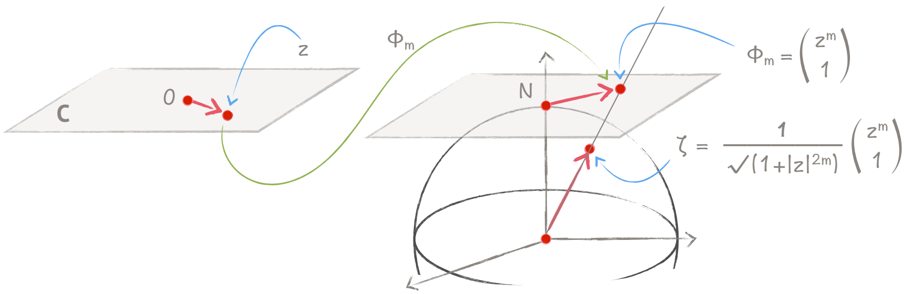


Figure 27. The waterdrop Orbifold.

• If $\zeta(r_0) = \mathrm{N}$, then there exists a small ball $\mathcal{B}$ centered at r_0 and a smooth parametrization z in $\mathbf{C}$ defined on $\mathcal{B}$ such that, for all $r \in \mathcal{B}$,

$$\zeta(r) = \frac{1}{\sqrt{1 + |z(r)|^{2m}}} \begin{pmatrix} z(r)^m \\ 1 \end{pmatrix}.$$

Note that this orbifold diffeology on S^2 is a *subdiffeology* of the standard diffeology of manifold, embedded in $\mathbf{R}^3$.

Note also that how it is possible to multiply the number of conical points on the sphere.

170. Example: The hedgehog. After the cone which has a unique singularity, of conic type and structure group U_m, at the north pole, it is not hard to imagine many other examples based on the construction of previous two: a sphere, or a plane, with as many different singular conic (or not) points we wants. The fact that, contrarily to manifolds, orbifolds may have a rich set of smooth local invariants, permits to build easily more different orbifold strutures on the same underlying space. In our examples above, we just picked up a diffeology finer than the standard manifold diffeology which happens to be an orbifold diffeology.

In particular, we have seen earlier that the quotient space of a disc by U_m is equivalent to the same disc but equipped with a finer diffeology. It is then easy to extract from a manifold diffeology, a finer orbifold diffeology with as many conic singularities as we want, with arbitrary

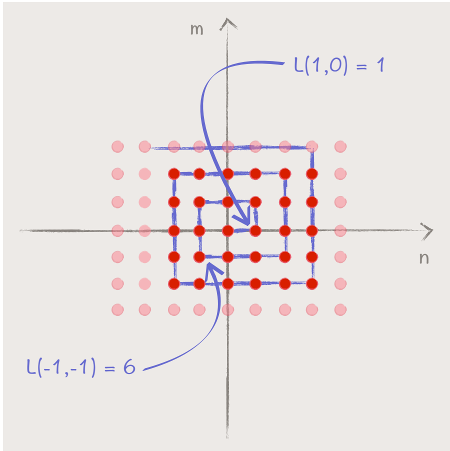


Figure 28. The function $L(n, m)$.

structure groups. Let's build a simple example: Consider the plane $\mathbf{C}$ and let $L : \mathbf{Z} + i\mathbf{Z} \to \mathbf{N}$ be the bijection described in Figure 28.

Then, let us define the following diffeology, with parametrizations $\zeta : U \to \mathbf{C}$ such that, for all $r_0 \in U$:

(1) if $\zeta(r_0) \notin \mathbf{Z} + i\mathbf{Z}$, then there exists a small ball $\mathcal{B}$ centered at r_0 such that $\zeta \upharpoonright \mathcal{B}$ is smooth.
(2) If $\zeta(r_0) = n + im$, with $n, m \in \mathbf{Z}$, then there exists a small ball $\mathcal{B}$ centered at r_0 and a smooth parametrization z in $\mathbf{C}$ defined on $\mathcal{B}$ such that, for all $r \in \mathcal{B}$,

$$\zeta(r) = n + im + z(r)^{1+L(n,m)}.$$

In this example, the integer points $n + im \in \mathbf{C}$ are conic with cyclic groups, all different, equal to $\mathbf{Z}_{1+L(n,m)}$.

It should be noted that, with all the transition functions, a description of this orbifold using the original Satake's defining families, would be, for the least, laborious. We can appreciate, on this example, the simplification brought by the diffeological approach.

171. Smooth Maps Between Orbifolds. Let us continue with the cone orbifold $\mathcal{C}_m = \mathbf{C}/\mathcal{U}_m$. Let $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } r > 1 \text{ or } r = 0 \\ e^{-1/r}\rho_n(r)(r,0) & \text{if } \frac{1}{n+1} < r \leq \frac{1}{n} \text{ and } n \text{ is even} \\ e^{-1/r}\rho_n(r)(x,y) & \text{if } \frac{1}{n+1} < r \leq \frac{1}{n} \text{ and } n \text{ is odd,} \end{cases}$$

where $r = \sqrt{x^2+y^2}$ and ρ_n is a function vanishing flatly outside the interval $]1/(n+1), 1/n[$ and not inside, see Figure 29. Remark now

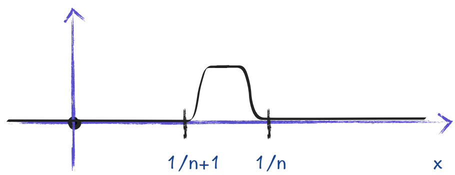


Figure 29. The function ρ_n.

that

$$f(AX) = h_X(A)f(X), \quad \text{with} \quad X \in \mathbf{R}^2 \quad \text{and} \quad A \in SO(2).$$

On the annulus

$$\frac{1}{n+1} < r \leq \frac{1}{n}, \quad \text{with} \quad \begin{cases} h_X(A) = 1_{\mathbf{R}^2} & \text{if } n \text{ is even, and} \\ h_X(A) = A & \text{if } n \text{ is odd.} \end{cases}$$

Now, the function f descends onto a smooth map φ from the cone orbifold $\mathcal{C}_m$ to itself. In particular because the homomorphism h_X flips from the identity to trivial on any successive anulus, φ has no local equivariant smooth lifting.

This is a big difference with diffeomorphisms for which it is proven that the stabilizer of one point is locally mapped equivariantly into the stabilizer of the image by a homomorphism.

$$\begin{array}{ccc} \mathbf{R}^2 & \xrightarrow{\ f\ } & \mathbf{R}^2 \\ \text{class}_m \downarrow & & \downarrow \text{class}_m \\ \mathcal{C}_m = \mathbf{R}^2/\mathcal{U}_m & \xrightarrow[\varphi]{} & \mathcal{C}_m = \mathbf{R}^2/\mathcal{U}_m \end{array}$$

This example is the very illustration of the unsuccessful attempt to define smooth maps between orbifolds as locally equivariant maps, on the level of local symmetry group, and that answer Satake footnote in [**Sat57**, page 469],

> "*The notion of $\mathcal{C}^\infty$-map thus defined is inconvenient in the point that a composite of two $\mathcal{C}^\infty$-maps defined in a different choice of defining families is not always a $\mathcal{C}^\infty$ map.*"

Embedding of orbifolds into a category such as {Diffeology} could have solved this question. The existence of good smooth maps between orbifolds is crucial for having a covariant satisfactory theory of orbifolds.

41. Equivalence between V-manifolds and D-orbifolds

172. V-manifolds are D-orbifolds. Let M be a Hausdorff topological space. A defining family $\mathcal{F}$ on M determines a diffeology of orbifold. Namely, the diffeology generated by the parametrizations $\varphi \colon \tilde{U} \to U$, for all $(\tilde{U}, G, \varphi) \in \mathcal{F}$.

173. Equivalence of defining families. Let M be a Hausdorff topological space. If two defining families $\mathcal{F}$ and $\mathcal{F}'$ on M generate the same diffeology then they are equivalent. More precisely, the union $\mathcal{F}'' = \mathcal{F} \cup \mathcal{F}'$ is a defining family.

174. D-orbifolds are V-manifolds. Conversely, let X be a D-orbifold, then equip X with the D-topology (assumed Hausdorff). Let $\mathcal{A}$ be an atlas of X, the strict generating family of the atlas $\mathcal{A}$ is a defining family in the sense of Satake, and equip X with a structure of V-manifold for its D-topology.

Two different atlases $\mathcal{A}$ and $\mathcal{A}'$ give equivalent Satake defining families, essentially because they are sub-atlases of the maximal atlas.

175. Equivalence of definitions. The two constructions above are, up to an equivelence, inverse one from each other.

Note. Actually, Satake defined only what we call *reflexion free V-manifolds*. But there was no technical obstacles to extend the definition to any V-manifold.

42. Internal structure of a D-orbifold

176. The groupoid G of germs of local diffeomorphisms. Let X be any diffeological space. We define the groupoid $\mathbf{G}$ of *germs of local diffeomorphisms* of X as follow:

$$\begin{cases} \mathrm{Obj}(\mathbf{G}) &= \mathrm{X}, \\ \mathrm{Mor}(\mathbf{G}) &= \{ \mathrm{germ}(\varphi)_x \mid \varphi \in \mathrm{Diff}_{\mathrm{loc}}(\mathrm{X}) \text{ and } x \in \mathrm{dom}(\varphi)\}. \end{cases}$$

The *source* maps, the *target* maps and the composition of germs of local diffeomorphisms are defined as follows:

$$\begin{cases} \mathrm{src}(\mathrm{germ}(\varphi)_x) = x, \quad \mathrm{trg}(\mathrm{germ}(\varphi)_x) = \varphi(x). \\ \mathrm{germ}(\varphi)_x \cdot \mathrm{germ}(\varphi')_{x'} = \mathrm{germ}(\varphi' \circ \varphi)_x, \text{ with } x' = \varphi(x). \end{cases}$$

The pseudogroup of local diffeomorphisms of X is equipped with the functional diffeology of pseudogroup, that is, the diffeology of local smooth map for the pairs (f, f^{-1}), where $f \in \mathrm{Diff}_{\mathrm{loc}}(\mathrm{X})$. Let then define the germ map by:

$$\begin{cases} \mathfrak{G} = \{(\varphi, x) \mid \varphi \in \mathrm{Diff}_{\mathrm{loc}}(\mathrm{X}) \text{ and } x \in \mathrm{dom}(\varphi)\}. \\ \mathrm{germ} : (\varphi, x) \mapsto \mathrm{germ}(\varphi)_x. \end{cases}$$

We equip $\mathrm{Mor}(\mathbf{G})$ with the *pushforward* of the diffeology of $\mathfrak{G}$ by the map germ. That makes $\mathbf{G}$ a diffeological groupoid. That means essentially that:

(a) the multiplication and the inversion are smooth maps

(b) and the inclusion $\mathrm{Obj}(\mathbf{G}) \hookrightarrow \mathrm{Mor}(\mathbf{G})$ by the identities is an induction.

177. The structure groupoid of an orbifold. Let X be an n-orbifold, $\mathcal{A}$ be an atlas, $\mathcal{F}$ be the strict generating family over $\mathcal{A}$, $\mathcal{N}$ be the nebula and ev be the evaluation map, that is:

$$\mathcal{N} = \coprod_{F \in \mathcal{F}} \mathrm{dom}(F) \quad \text{and} \quad \mathrm{ev} : \mathcal{N} \to X \quad \text{with} \quad \mathrm{ev}(F, r) = F(r).$$

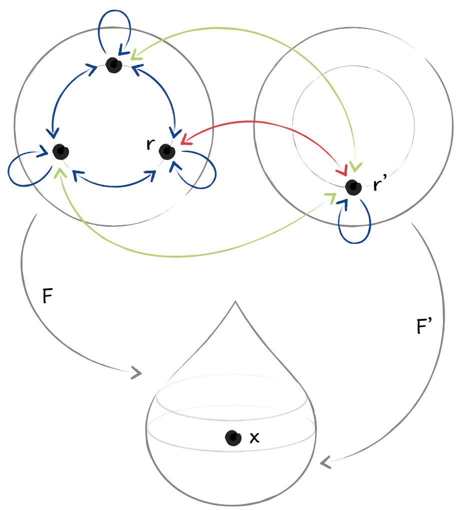


Figure 30. The Groupoid of the Teardrop.

We call *Structure groupoid* $\mathbf{G}$ of the orbifold X the subgroupoid of the groupoid of germs of local diffeomorphisms of $\mathcal{N}$ that descends on the identity of X along ev. That is,

$$\mathrm{Mor}_{\mathbf{G}}((F, r), (F', r')) = \left\{ \mathrm{germ}(\varphi)_r \,\middle|\, \begin{array}{l} \varphi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbf{R}^n), r' = \varphi(r) \\ F' \circ \varphi = F \restriction \mathrm{dom}(\varphi) \end{array} \right\}$$

Note. In order to show the dependency of the structure groupoid with respect to the atlas $\mathcal{A}$ we need the two following lemma.

178. Lifting the identity. Let $\mathcal{Q} = \mathbf{R}^n/\Gamma$. Consider a local smooth map F from $\mathbf{R}^n$ to itself, such that $\text{class} \circ \text{F} = \text{class}$. In other words, F is a local lifting of the identity on $\mathcal{Q}$. Then,

$$\begin{array}{ccc} \mathbf{R}^n \supset \tilde{\text{U}} & \xrightarrow{\quad \text{F} \quad} & \mathbf{R}^n \\ & {}_{\text{class}}\searrow \quad \swarrow_{\text{class}} & \\ & \mathcal{Q} & \end{array}$$

Theorem. *F is locally equal to some group action* $\text{F}(r) =_{\text{loc}} \gamma \cdot r$*, where* $\gamma \in \Gamma$*.*

Proof. Let us assume first that F is defined on an open ball $\mathcal{B}$. Then, for all r in the ball, there exists a $\gamma \in \Gamma$ such that $\text{F}(r) = \gamma \cdot r$. Next, for every $\gamma \in \Gamma$, let

$$\text{F}_\gamma : \mathcal{B} \to \mathbf{R}^n \times \mathbf{R}^n \quad \text{with} \quad \text{F}_\gamma(r) = (\text{F}(r), \gamma \cdot r).$$

Let $\Delta \subset \mathbf{R}^n \times \mathbf{R}^n$ be the diagonal and let us consider

$$\Delta_\gamma = \text{F}_\gamma^{-1}(\Delta) = \{r \in \mathcal{B} \mid \text{F}(r) = \gamma \cdot r\}.$$

Lemma 1. *There exist at least one* $\gamma \in \Gamma$ *such that the interior* $\mathring{\Delta}_\gamma$ *is non-empty.*

◀ Indeed, since F_γ is smooth (thus continuous), the preimage Δ_γ by F_γ of the diagonal is closed in $\mathcal{B}$. However, the union of all the preimages $\text{F}_\gamma^{-1}(\Delta)$ — when γ runs over Γ — is the ball $\mathcal{B}$. Then, $\mathcal{B}$ is a finite union of closed subsets. According to Baire's theorem, there is at least one γ such that the interior $\mathring{\Delta}_\gamma$ is not empty. ▶

Lemma 2. *The union* $\mathring{\Delta}_\Gamma = \cup_{\gamma \in \Gamma} \mathring{\Delta}_\gamma$ *is an open dense subset of* $\mathcal{B}$*.*

◀ Indeed, let $\mathcal{B}' \subset \mathcal{B}$ be an open ball. Let us denote with a prime the sets defined above but for $\mathcal{B}'$. Then, $\Delta'_\gamma = (\text{F}_\gamma \restriction \mathcal{B}')^{-1}(\Delta) = \Delta_\gamma \cap \mathcal{B}'$, and then $\mathring{\Delta}'_\gamma = \mathring{\Delta}_\gamma \cap \mathcal{B}'$. Thus, $\mathcal{B}' \cap \mathring{\Delta}_\Gamma = \mathcal{B}' \cap (\cup_{\gamma \in \Gamma} \mathring{\Delta}_\gamma) = \cup_{\gamma \in \Gamma} \mathring{\Delta}'_\gamma$, which is not empty for the same reason that $\cup_{\gamma \in \Gamma} \mathring{\Delta}_\gamma$ is not empty. Therefore, $\mathring{\Delta}_\Gamma$ is dense. ▶

In conclusion: the tangent linear map $\text{D}(\text{F}) : \mathcal{B} \to \text{GL}(n, \mathbf{R})$ is smooth, then continuous, thus $\text{D}(\text{F})^{-1}(\Gamma)$ is closed. But, $\mathring{\Delta}_\Gamma$, which is an open dense subset of $\mathcal{B}$, is contained in $D(F)^{-1}(\Gamma)$. Hence, $\mathcal{B}$ is contained in $D(F)^{-1}(\Gamma)$ (its own closure) which is contained in $\mathcal{B}$. Thus, $D(F)^{-1}(\Gamma) = \mathcal{B}$. Then, since $\mathcal{B}$ is connected, $D(F)(\mathcal{B}) \subset \Gamma$ is connected. But $\Gamma \subset GL(n, \mathbf{R})$ is discrete, then $D(F)(\mathcal{B}) = \{\gamma\}$, for some $\gamma \in \Gamma$. ▶

179. **Lifting local diffeomorphisms.** Let $\mathcal{Q} = \mathbf{R}^n/\Gamma$ and $\mathcal{Q}' = \mathbf{R}^{n'}/\Gamma'$, Then,

Theorem. *Every local smooth lifting $\tilde{f}$ of any local diffeomorphism f, from $\mathcal{Q}$ to $\mathcal{Q}'$, is necessarily a local diffeomorphism, from $\mathbf{R}^n$ to $\mathbf{R}^{n'}$. In particular $n = n'$. Moreover, let*

$$x \in \mathrm{dom}(f), \ x' = f(x)$$

and $r, r' \in \mathbf{R}^n$ such that

$$\mathrm{class}(r) = x \quad \textit{and} \quad \mathrm{class}(r') = x'.$$

Then, the local lifting $\tilde{f}$ can be chosen such that

$$\tilde{f}(r) = r'.$$

Proof. Let the local diffeomorphism f be defined on U with values in U'. By definition of local diffeomorphism, they are both open for the D-topology. Then $\tilde{U} = \mathrm{class}^{-1}(U)$ is open in $\mathbf{R}^n$. Since the composite $f \circ \mathrm{class} \colon \tilde{U} \to U'$ is a plot in $\mathcal{Q}'$, for all $r \in \tilde{U}$ there exists a smooth local lifting $\tilde{f} \colon \tilde{V} \to \mathbf{R}^{n'}$, defined on an open neighborhood of r, such that $\mathrm{class}' \circ \tilde{f} = f \circ \mathrm{class} \restriction \tilde{V}$.

$$\begin{array}{ccc}
\mathbf{R}^n \supset \tilde{U} \supset \tilde{V} & \xrightarrow{\ \tilde{f}\ } & \mathbf{R}^{n'} \\
\Big\downarrow \mathrm{class} & & \Big\downarrow \mathrm{class}' \\
\mathcal{Q} \supset U & \xrightarrow[\ f\ ]{} & \mathcal{Q}'
\end{array}$$

Let $x = \mathrm{class}(r)$, $x' = f(x)$, $r' = \tilde{f}(r)$, and then $x' = \mathrm{class}'(r')$.

Next, let $\tilde{U}' = \mathrm{class}'^{-1}(U')$. Since the composite $f^{-1} \circ \mathrm{class}'$ is a plot in $\mathcal{Q}$, there exists a smooth lifting $\hat{f} \colon \tilde{V}' \to \mathbf{R}^n$, defined on an open neighborhood of r', such that $\text{class} \circ \hat{f} = f^{-1} \circ \text{class}' \restriction \tilde{V}'$. Let $r'' = \hat{f}(r')$, which can be different from r.

$$\begin{array}{ccc}
\mathbf{R}^n & \xleftarrow{\quad \hat{f} \quad} & \tilde{V}' \subset \tilde{U}' \subset \mathbf{R}^{n'} \\
\text{class}\big\downarrow & & \big\downarrow\text{class}' \\
Q & \xleftarrow[\quad f^{-1} \quad]{} & U' \subset Q'
\end{array}$$

Now, we consider the composite $\hat{f} \circ \tilde{f} \colon \tilde{W} \to \mathbf{R}^n$, where $\tilde{W} = \tilde{f}^{-1}(\tilde{V}')$ is a non-empty open subset of $\mathbf{R}^n$ since it contains r. Moreover, $\hat{f} \circ \tilde{f}(r) = r''$. It also satisfies $\text{class} \circ (\hat{f} \circ \tilde{f}) = \text{class}$. Indeed, $\text{class} \circ (\hat{f} \circ \tilde{f}) = (\text{class} \circ \hat{f}) \circ \tilde{f} = (f^{-1} \circ \text{class}') \circ \tilde{f} = f^{-1} \circ (\text{class}' \circ \tilde{f}) = f^{-1} \circ (f \circ \text{class}) = (f^{-1} \circ f) \circ \text{class} = \text{class}$. Thus, thanks to § 178, there exists, locally, $\gamma \in \Gamma$ such that $\hat{f} \circ \tilde{f} = \gamma \restriction \tilde{W}$. By the way, $r'' = (\hat{f} \circ \tilde{f})(r) = \gamma \cdot r$. Let $\bar{f} = \gamma^{-1} \circ \hat{f}$, then: $\text{class} \circ \bar{f} = \text{class} \circ \gamma^{-1} \circ \hat{f} = \text{class} \circ \hat{f} = f^{-1} \circ \text{class}'$, and $\bar{f}$ is still a local lifting of f^{-1}. Thus $\bar{f} \circ \tilde{f} = \mathbf{1}_{\tilde{W}}$, that is, $\bar{f} = \tilde{f}^{-1} \restriction \tilde{W}$. We conclude that, around r, $\tilde{f}$ is a local diffeomorphism. Now, if we consider any another point r''' over x', there exists γ' such that $\gamma' \cdot r' = r'''$; changing $\tilde{f}$ to $\gamma' \circ \tilde{f}$ and $\bar{f}$ to $\bar{f} \circ \gamma'^{-1}$, we get $\tilde{f}(r) = r'''$, and $\tilde{f}$ and $\bar{f}$ still remain inverse of each other. Thus, for any $r \in \mathbf{R}^n$ over x and any $r' \in \mathbf{R}^n$ over $x' = f(x)$, we can locally lift f to a local diffeomorphism $\tilde{f}$ such that $\tilde{f}(r) = r'$. ▶

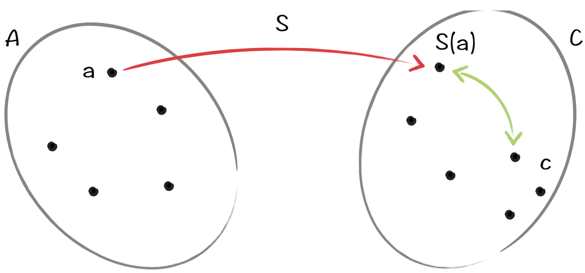


Figure 31. Equivalence of categories.

<u>**180. Equivalence between categories, groupoids.**</u> Let A and C be two categories. Let us recall that, according to [McL71, Chap.4 § 4 Thm. 1],

a *functor*

$$S : A \to C$$

is an *equivalence of categories* if and only if,

(1) S is full and faithful,
(2) each object c in C is isomorphic to S(a) for some object a in A.

If A and C are groupoids, the last condition means that,

(2') for each object c of C, there exists an object a of A and an arrow from S(a) to c.

In other words: Let the *transitivity-components* of a groupoid be the maximal full subgroupoids such that each object is connected to any other object by an arrow. The functor S is an equivalence of groupoids if

(1) it is full and faithful,
(2) it descends surjectively on the set of transitivity-components.

$$\begin{array}{ccc} A & \xrightarrow{\quad S \quad} & C \\ {\scriptstyle \mathrm{comp}}\downarrow & & \downarrow{\scriptstyle \mathrm{comp}} \\ \mathrm{Comp}(A) & \xrightarrow[S_*]{} & \mathrm{Comp}(C) \end{array}$$

181. Equivalence of structure-groupoids. Consider an n-orbifold X. Let $\mathcal{A}$ be an atlas, let $\mathcal{F}$ be the associated strict generating family, let $\mathcal{N}$ be the nebula of $\mathcal{F}$ and let $\mathbf{G}$ the associated structure groupoid.

Proposition. *The fibers of the subduction* $\mathrm{ev} : \mathrm{Obj}(\mathbf{G}) \to X$ *are exactly the transitivity-components of* $\mathbf{G}$*. In other words, the space of transitivity components of the groupoid* $\mathbf{G}$ *associated with any atlas of the orbifold* X*, equipped with the quotient diffeology, is the orbifold itself.*

Theorem. *Different atlases of* X *give equivalent structure groupoids. The structure groupoids associated with diffeomorphic orbifold are equivalent.*

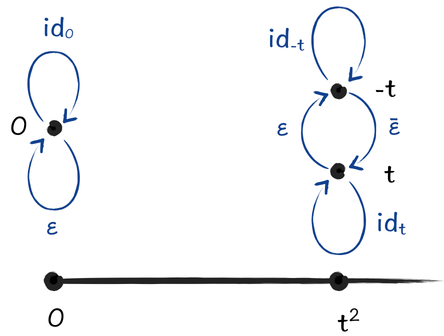


Figure 32. The Groupoid of Δ_1.

In other words, the equivalence class, in the sense of categories, of the structure groupoids of a orbifold is a diffeological invariant.

☞ Proof. The proof of this theorem is based on the previous two propositions § 178 and § 179.

Let us start by proving the proposition. Let

$$\mathrm{F}\colon \mathrm{U} \to \mathrm{X} \quad \text{and} \quad \mathrm{F}'\colon \mathrm{U}' \to \mathrm{X}'$$

be two generating plots from the strict family $\mathcal{F}$, and $r \in \mathrm{U} \subset \mathbf{R}$ and $r' \in \mathrm{U}' \subset \mathbf{R}'$. Assume that

$$\mathrm{ev}(\mathrm{F}, r) = \mathrm{ev}(\mathrm{F}', r') = x, \quad \text{that is} \quad , x = \mathrm{F}(r) = \mathrm{F}'(r').$$

Note that

$$\mathrm{F} = f \circ \mathsf{class} \restriction \mathrm{U} \quad \text{and} \quad \mathrm{F}' = f' \circ \mathsf{class}' \restriction \mathrm{U}',$$

where $f, f' \in \mathcal{A}$. Then, let

$$\psi = f'^{-1} \circ f \quad \text{with} \quad \psi\colon f^{-1}(f'(\mathrm{U}')) \to \mathrm{U}',$$

is a local diffeomorphism that maps

$$\xi = f(\mathsf{class}(r)) \quad \text{to} \quad \xi' = f'(\mathsf{class}'(r')).$$

Then, according to § 179:

(1) $n = n'$,

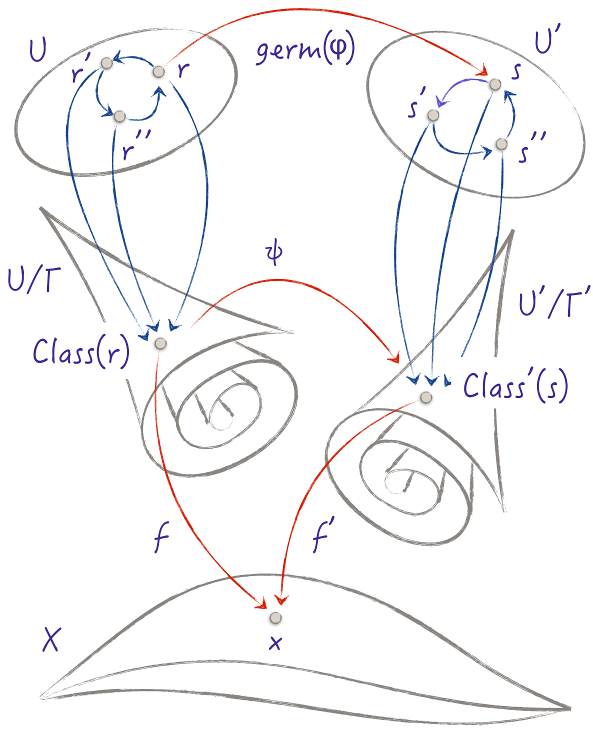


Figure 33. Lifting the identity.

(2) there exists a local diffeomorphism φ of $\mathbf{R}^n$, lifting locally ψ and mapping r to r'. That is

$$\text{class}' \circ \phi =_{\text{loc}} \psi \circ \text{class} \quad \text{and} \quad \psi(r) = r'.$$

Its germ realizes a morphism (an arrow) of the groupoid $\mathbf{G}$ connecting (F, r) to (F', r'), which are then on the same transitivity component:

$$\mathrm{F}(r) = \mathrm{F}'(r') \Rightarrow \text{comp}(\mathrm{F}, r) = \text{comp}(\mathrm{F}', r').$$

Of course, when $\mathrm{F}(r) \neq \mathrm{F}'(r')$ there cannot be an arrow, by definition. Therefore, the fibers of the evaluation map are the transitive components of the structure groupoid $\mathbf{G}$ of the orbifold.

Now, for the theorem: let $\mathcal{A}$ and $\mathcal{A}'$ be two atlases of X and consider

$$\mathcal{A}'' = \mathcal{A} \coprod \mathcal{A}'.$$

With an obvious choice of notation:

$$\mathrm{Obj}(\mathbf{G}'') = \mathrm{Obj}(\mathbf{G}) \coprod \mathrm{Obj}(\mathbf{G}'),$$

and $\mathbf{G}''$ contains naturally $\mathbf{G}$ and $\mathbf{G}'$ as full subgroupoids. The question then is: how does the adjunction of the crossed arrows between $\mathbf{G}$ and $\mathbf{G}'$ change the distribution of transitivity-components? According to the previous proposition, it changes nothing since, for $\mathbf{G}$, $\mathbf{G}'$ or $\mathbf{G}''$, the set of transitivity-components are always exactly the fibers of the respective subductions ev. In other words, the set of groupoid components is always X, for any atlas of X. Thus $\mathbf{G}$ and $\mathbf{G}'$ are equivalent to $\mathbf{G}''$, therefore $\mathbf{G}$ and $\mathbf{G}'$ are equivalent. ▶

43. Quasifolds as diffeologies

The notion de *quasifold* has been proposed by Elisa Prato in [**EP01**], it extend the notion of orbifold. The original definition, which has been modified once or twice, has been revisited by diffeology as follow:

182. Definition. *A diffeological space* X *is said to be a n-quasifold if it is locally diffeomorphic everywhere to a quotient* $\mathbf{R}^n/\Gamma$*, where* Γ *is a countable subgroup of* $\mathrm{Aff}(\mathbf{R}^n)$.

The main difference is that the group Γ *can be infinite, but countable.*

Example 1. The first example would certainly be the irrational torus $T_\alpha = \mathbf{R}/(\mathbf{Z}+\alpha\mathbf{Z})$ *with* $\alpha \in \mathbf{R}-\mathbf{Q}$*. Also quotient of the* 2*-torus by the irrational solenoid* $\mathcal{S}_\alpha$*. Here the subgroup is strictly affine.*

Example 2. One can think also of the irrational cone *or* quasicone*:*

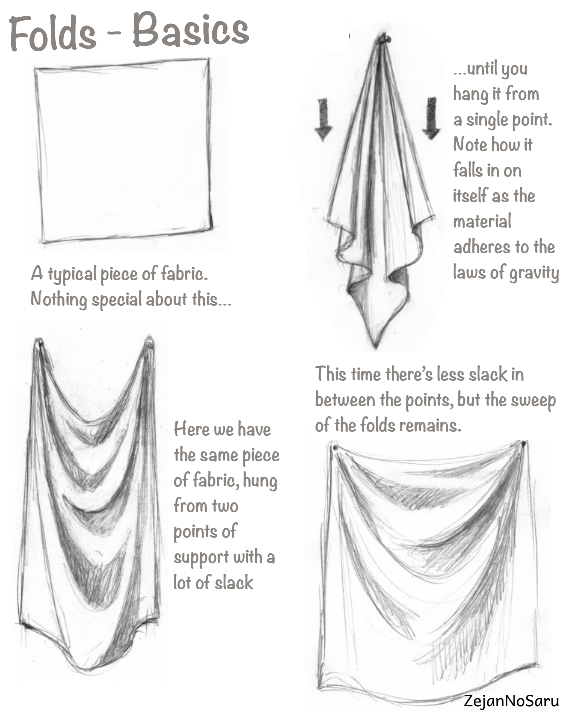
$$\mathcal{C}_\alpha = \mathbf{C}/\{e^{2i\pi k\alpha}\}_{k\in\mathbf{Z}},$$

where $\alpha \in \mathbf{R}-\mathbf{Q}$.

As a curiosity we have this triangle of subductions:

$$\begin{array}{ccc} & \mathbf{C} & \\ {}^{\text{class}}\swarrow & & \searrow^{|\cdot|^2} \\ \mathcal{C}_\alpha & \xrightarrow[\pi]{} & \Delta_2 \end{array}$$

where class : $\mathbf{C} \to \mathcal{C}_\alpha$ *is the standard projection from the space to its quotient, and* $|\cdot|^2 : z \mapsto |z|^2$ *is the projection from* $\mathbf{C}$ *to the quotient* $\mathbf{C}/\mathrm{U}(1)$ *which is equivalent to* $\Delta_2 = \mathbf{R}^2/\mathrm{O}(2)$. *Then, the map* $\pi : \mathcal{C}_\alpha \to \Delta_2$ *is a subduction since* class *and* $|\cdot|^2 = \pi \circ \text{class}$ *are both subductions* [**TB**, § 1.51]. *The preimages of* π *are all diffeomorphic to the irrational torus* T_α *except for* 0. *It would be an exercise probably to show that the local diffeomorphisms of* $\mathcal{C}_\alpha$ *has two orbits:* $\{0\}$ *and* $\mathcal{C}_\alpha - \{0\}$. *Note that the D-topology of* $\mathcal{C}_\alpha$ *is certainly, by density, the pullback of the D-topology of* Δ_2*, which coincides with the topology of the half-line* $[0, \infty[$*.s*



Symplectic Mechanics and Diffeology

I give, in this lecture, a short survey on symplectic mechanics. The basic foundations in the work of Lagrange at the end of the 18th century and the beginning of the 19th, and what it became in the middle of the 20th century with what it is called now: *symplectic mechanics*. We will discuss also how to extend these constructions to Diffeology, the main concepts.

There are a few different approaches to symplectic geometry in mechanics. At the beginning there is three papers from Joseph-Louis Lagrange in 1808, 1809 and 1810 [**Lag08, Lag09, Lag10**] :

1) *Sur la théorie des variations des éléments des planètes et en particulier des variations des grands axes de leurs orbites* (1808).

2) *Sur la théorie générale de la variation des constantes arbitraires dans tous les problèmes de la mécanique*(1809).

3) *Second mémoire sur la théorie générale de la variation des constantes arbitraires dans tous les problèmes de la mécanique* (1810).

In these papers, Lagrange sets the first elements of what we can call "symplectic calculus". The question was the stability of the great axes of the planets, and Lagrange brought a simplification in the approximation computations of this time, in particular by Laplace and Poisson. I will not write down here the details of his work but I refer (for now) to the paper I wrote on the subject: *Les Origines du Calcul Symplectique chez Lagrange*, but in French [**PIZ98**].

That said, I will emphasize — in this overview on symplectic mechanics — the following points relative to the global structure of the spaces of solutions of dynamical (symplectic) systems.

(1) The presymplectic/symplectic framework.
(2) The symmetries of dynamical systems.
(3) Isolated mechanical/dynamical systems, the Galilean and Poincaré groups.
(4) The moment map associated to a group of symmetries.
(5) The conservation of the moment map (Noether-Souriau theorem).
(6) The Souriau cocycle and the barycentric decomposition.
(7) The elementary particles/systems and the classic spin.
(8) The "geometric quantization program", the prequantization.

These few points summarizes, I believe, the main progress on the global structure of dynamical systems made in the 20th century.

Of course, symplectic mechanics does not reduce to these chapters, other constructions like the behavior of hamiltonian vector fields, the geometric optics, reflexion, diffraction, caustics. . . The structure of the group of symplectomorphisms, etc. All these subjects are a part of modern symplectic mechanics. A whole year of lectures could hardly be enough to cover all the applications of symplectic mechanics.

44. The short approach to symplectic mechanics

The following presentation of the syplectic structure on the space of solutions of a dynamical (second order differential equation) system is due to Elie Cartan [**Car22**], followed by some authors, Galissot [**Gal52**], and especially Jean-Marie Souriau in his book "Structure des Systèmes Dynamique" [**Sou70**].

183. To drive out the denominators. Let us recall that a classical dynamical system in Galilean Mechanics, or Newtonian Mechanics, is described by an ordinary second order differential equation. For

example, for a material point:

$$m\frac{d^2x}{dt^2} = \mathrm{F}(x, v, t) \quad \text{where} \quad v = \frac{dx}{dt}.$$

The unknown here is the path $[t \mapsto x]$, where t is a real number defined on some interval and $x \in \mathbf{R}^3$.

Then we transform this second order system into a first order differential equation system:

$$m\frac{dv}{dt} = \mathrm{F}(x, v, t) \quad \text{and} \quad v = \frac{dx}{dt}.$$

Then we reinterpret the Newton equations by "driving out the denominators" :

$$mdv = \mathrm{F}(x, v, t)dt \quad \text{and} \quad dx = vdt.$$

We need here to explain this writing. Consider the subset

$$\mathrm{Y} \subset \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}$$

where F is defined. We call it the space of initial conditions, or following Souriau: the "evolution space". Let $y = (x, v, t)$ a point in Y, let us denote by dy a tangent vector to Y, at the point y, that is a vector of $\mathbf{R}^7$. We can imagine dy being the shortcut the derivative of a path $s \mapsto y$

$$dy = \frac{dy}{ds} \in \mathrm{T}_y\mathrm{Y} \quad \text{with} \quad dy = \begin{pmatrix} dx \\ dv \\ dt \end{pmatrix}.$$

Then, the equations of motion writes

$$mdv - \mathrm{F}(x, v, t)dt = 0 \quad \text{and} \quad dx - vdt = 0.$$

At this point we use a trick, and considering an other tangent vector

$$\delta y = \begin{pmatrix} \delta x \\ \delta v \\ \delta t \end{pmatrix} \in \mathrm{T}_y\mathrm{Y},$$

we define

$$\omega(dy, \delta y) = \langle mdv - \mathrm{F}dt, \delta x - v\delta t\rangle - \langle m\delta v - \mathrm{F}\delta t, dx - vdt\rangle.$$

Let us make some comment on the notation: first of all, F denotes at the same time the function F and the value of F at the point $y = (x, v, t)$. Second of all, the 2-form ω should rigorously be indexed by the point y where it is taken, but the vectors dy and δy contains already this information, so it is unnecessary to be redundant.

One can check now that:

(1) ω is a 2-form on Y.
(2) the vector $dy \simeq dy/ds$ satisfies the *equations of motion* if and only if it belongs to the kernel of ω.

The kernel of ω is defined by

$$dy \in \ker(\omega) \quad \Leftrightarrow \quad \omega(dy, \delta y) = 0 \quad \forall \delta y.$$

Therefore, the integral curves of the kernel distribution

$$y \mapsto \ker(\omega)$$

are the solutions of the Newton equations.

Proposition 1. *The space $\mathcal{M}$ of integral curves of the kernel distribution is a manifold, that can be, for some kind of force* F*, non Hausdorff. It is called the* space of motions*.*

I would like to emphasize the fact the a solution of the differential equation, in this approach, is an integral curve of the distribution $y \mapsto \ker(\omega)$, that is, a subset of Y, the graph of some curve $t \mapsto (x, v, t)$. So the space of motions is indeed a set of subspaces of Y, and this set of subspaces is equipped with a structure of manifold.

And beware to not confuse the motion with the trajectory of the motion. The schedule of the trajectory is a part of the motion. For example, a circular motion has a circle for trajectory but its motion is a line, precisely an helix. That is why it is not enough to know the route of the bus to take advantage of it, you have to know when it stops in front of your door. The timetable is an element of the movement.

Proposition 2. *The restrictions* $(x, v) \mapsto \mathrm{class}_t(x, v)$*, for all* $t \in \mathbf{R}$*, made a canonical atlas of the manifold* $\mathcal{M}$*.*

Note that these canonical charts class_t are called *Darboux charts* because the symplectic form takes the canonical expression,

$$\omega_t = m \sum_{i=1}^{3} dx_i \wedge dv_i.$$

Proposition 3. *If the force* F *is the gradient of some* potential ϕ*, actually* $F = -\mathrm{grad}(\phi)$*, then the* 2*-form* ω *is closed*

$$d\omega = 0,$$

and descends on the quotient space $\mathcal{M}$ *into a symplectic* 2*-form.*

In that case, the 2-form ω is the exterior derivative of the so-called *Cartan form*

$$\lambda(\delta y) = m\langle v, \delta x\rangle - h\,\delta t \quad \text{with} \quad h = \frac{1}{2}m\|v\|^2 + \phi.$$

Definition. *We recall that a symplectic form on a manifold is a non degenerate closed* 2*-form.*

45. Presymplectic and symplectic manifolds

184. Presymplectic form. Let M be a manifold, a closed 2-form ω on M is said to be *presymplectic* if its kernel has a constant dimension on M [**Sou70**]. The vector distribution

$$y \mapsto \ker(\omega)$$

is called the *characteristic distribution*. Thanks to the Frobenious theorem that states that for every differential form ω, the characteristic distribution

$$y \mapsto \ker(d\omega) \cap \ker(\omega)$$

is integrable, since $\ker(d\omega)_y = T_y M$, the characteristic distribution of a closed form $y \mapsto \ker(\omega)$ is integrable.

The leaves of the integral submanifolds of the characteristic distribution are called *characteristics* of the distribution, and the resulting foliation is called the *characteristic foliation*.

The characteristics of the presymplectic form ω are the connected submanifolds $F \subset M$ such that at each point $y \in F$,

$$T_y F = \ker(\omega).$$

185. Symplectic dynamical systems. In *symplectic mechanics*, a dynamical system is defined as a presymplectic manifold (M, ω). By analogy with the case of a particle, we call *motions* of the system the characteristics of the presymplectic form. The *space of motions*, which plays an important role in mechanics, is then the set of all characteristics.

Note. There is no reason that, in general, the space of motions inherits a manifold structure by quotient, and it does not. That can be an obstacle in ordinary differential geometry, but not from the general diffeology point of view, where any case can be dealt with. The space of characteristic of a presymplectic manifold can always be equipped with the quotient diffeology, for which the usual diffeological tools continue to work. We denote this space by

$$\mathcal{M} = M/\ker(\omega).$$

186. Symplectic form. A closed 2-form ω on a manifold M is said to be symplectic if it is *non degenerate*, that is, if its kernel is reduced to $\{0\}$

$$\omega \text{ symplectic} \quad \Leftrightarrow \quad d\omega = 0 \text{ and } \ker(\omega) = 0.$$

A symplectic manifold is then a presymplectic manifold with a trivial characteristic foliation:

$$F_y = \{y\}.$$

Proposition. Consider a presymplectic manifold (M, ω), if the quotient space $\mathcal{M} = M/\ker(\omega)$ is a manifold, it inherits a symplectic form for which ω is the pullback.

That is why symplectic geometry is important in physics:

(1) Conservative dynamical systems are identified as presymplectic dynamical systems, for which the characteristic leaves are the solutions, or motions, of the system.
(2) The space of solutions of a conservative dynamical system, if it is a manifold itself, inhertis a symplectic structure.

The word "conservative" will be explained later.

187. Example: The geodesics trajectories on the Sphere. Consider the sphere S^2. Let US^2 be the unitary tangent bundle:

$$US^2 = \{(x, u) \in \mathbf{R}^3 \times \mathbf{R}^3 \mid \|x\| = \|u\| = 1 \text{ and } \langle x, u\rangle = 0\}.$$

Define on US^2 the 1-form

$$\lambda_y(\delta y) = \langle u, \delta x\rangle, \quad \text{with} \quad \begin{cases} y &= (x, u) \in US^2 \\ \delta y &= (\delta x, \delta u) \in T_y US^2. \end{cases}$$

let $\omega = d\lambda$, precisely

$$\omega_y(\delta y, \delta' y) = \langle \delta u, \delta' x\rangle - \langle \delta' u, \delta x\rangle.$$

One can check that ω is presymplectic and its charactetistic distribution is defined by

$$\frac{dy}{ds} \in \ker\omega \quad \Leftrightarrow \quad \frac{dx}{ds} = \alpha u \ \text{ and } \ \frac{du}{ds} = -\alpha x,$$

for all $\alpha \in \mathbf{R}$. The solutions of this sytem are great circles described by the *law of motion*

$$x(s) = e^{sj(\ell)} x_0 \ \text{ and } \ u(s) = e^{sj(\ell)} u_0 \ \text{ with } \ \ell = x_0 \wedge u_0.$$

Note that $\ell = x \wedge u$ is constant on the solutions, it is called the *kinetic momentum*. It is a particular case of the general moment map theory we talk later in the following.

Thus, the map

$$y = (x, u) \mapsto \ell = x \wedge u$$

from Y to S^2 realizes the quotient space

$$Y/\ker(\omega) \simeq S^2.$$

And the symplectic structure of the space of motions is equal, up to some constant, to the standard *area form*

$$\mathrm{Surf}_\ell(\delta\ell, \delta'\ell) = \langle \ell, \delta\ell \wedge \delta'\ell \rangle.$$

This example is a special case of the general construction of the set of *geodesic trajectories*, which are equipped with a symplectic structure as soon as it is a manifold.

188. Example: The geodesics trajectories on the torus. The geodesic trajectories of the 2-torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ are the projections of the affine lines in $\mathbf{R}^2$. Consider the projection which associates with each geodesic trajectory Δ, its direction $u \in S^1$.[1] This is a surjection. We can write it

$$\pi \colon \mathfrak{G}_{\mathrm{traj}} \to S^1 \quad \text{with} \quad \pi(\Delta) = u.$$

Now, the fiber over the direction $u \in S^1$ are all the projections on the 2-torus T^2 of the affine lines in $\mathbf{R}^2$, parallel to the unique line of direction u, and passing through the origin. Depending on the orientation, these lines cut the axis *oy* or the axis *ox* according the action of $\mathbf{Z} \oplus \mathbf{Z}$

$$(n, m) \colon y \mapsto y + n + \tau m,$$

with

$$u = (\alpha, \beta), \text{ with } \alpha^2 + \beta^2 = 1, \text{ and } \tau = \frac{\beta}{\alpha}.$$

Thus,

$$\pi^{-1}(u) \simeq T_\tau,$$

where T_τ is the torus of slope τ. If τ is rational then T_τ is diffeomorphic to the torus S^1, otherwise it is not a manifold but a diffeological space called the *irrational torus* (of slope τ) [DI83].

So, in the case of the torus the space of geodesic trajectories is not a manifold but at least a diffeological space. We prove further in the book that the canonical symplectic form on the space of geodesic curves (which is always a manifold) descends, in some sense, to the quotient $\mathfrak{G}_{\mathrm{traj}}$, as a differential closed 2-form, according to the definition in diffeology.

[1] We consider first the oriented geodesic trajectories.

That example shows in particular the necessity to enlarge the category of manifolds if we want to describe a bigger class of dynamical systems than usually.

189. Darboux theorem. An important theorem due to the mathematician Jean Gaston Darboux makes explicit the local nature of presymplectic and symplectic manifolds. Let (M, ω) be a presymplectic manifold of rank $2n$; the rank is defined by

$$\operatorname{rank}(\omega) = \dim(\mathrm{M}) - \dim(\ker(\omega)).$$

The rank of a 2-form is always even, let $\dim(\mathrm{M}) = 2n + k$. Then:

There exists an atlas $\mathcal{A}$ of charts such that, in any chart $\mathrm{F} \in \mathcal{A}$, the form ω can be identified with the matrix, if $k > 0$:

$$\begin{pmatrix} 0_n & -1_n & 0 \\ 1_n & 0_n & 0 \\ 0 & 0 & 0_k \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$$

in the symplectic case $k = 0$.

This is an important, even crucial, theorem with an enormous set of applications. In a few words: a symplectic structure is *flat*, always.

* There is no local invariants.
* All symplectic invariants are global.

46. Symmetries and moment map

The symmetries of a dynamical system, and its consequences, are certainly at the heart of symplectic mechanics.

190. Symmetries of a system. Let (M, ω) be a dynamical system, that is, a presymplectic manifold. We call a *symmetry* of the system any diffeomorphism $f \in \mathrm{Diff}(\mathrm{M})$ that preserves the presymplectic form ω. So, we define the largest group of symmetries of the system

$$\mathrm{Diff}(\mathrm{X}, \omega) = \{f \in \mathrm{Diff}(\mathrm{X}) \mid f^*(\omega) = \omega\}$$

When the form is symplectic, such a diffeomorphism is called a *symplectomorphism*.

Proposition. *Except for trivial cases, the group* $\mathrm{Diff}(X, \omega)$ *is infinite dimensional.*

Theorem. *On symplectic manifolds,* $\mathrm{Diff}(X, \omega)$ *is transitive on* M. *Actually* $\mathrm{Diff}(X, \omega)$ *is n-transitive if* $\dim(X) = 2n$ [**Boo69**].

191. Galilean mechanics. The history has produced three kind of Mechanics, each of them characterized by a group of transformations [**PIZ18**]. The *Aristotelean group*, the *Galilean group* and the *Poincaré group*. The Aristotelean mechanics has not been much developped.[2] The Galilean group is a 10-dimensional Lie group, made of matrices

$$m = \begin{pmatrix} A & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad A \in SO(3); b, c \in \mathbf{R}^3 \text{ and } e \in \mathbf{R}.$$

This group is associated with Galilean/Newtonian mechanics. It acts on $\mathbf{R}^3 \times \mathbf{R}$ by

$$\begin{pmatrix} A & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} Ar + bt + c \\ t + e \\ 1 \end{pmatrix}$$

Now we can anwser the Question:

What is a (Galilean) isolated system?

Principle *An isolated Galilean system is a presymplectic manifold* (M, ω) *with a symmetric action of the Galilean group.*

Later we will see that one adds the condition for this action to be Hamiltonian.

192. Relativity and the Poincaré Group. The group of Einstein Relativity is the Poincaré Group, that is, the group of affine transformations that preserve the quadratic form

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2,$$

with $(x, y, z, t) \in \mathbf{R}^3 \times \mathbf{R}$.

$$g : X \mapsto LX + C,$$

[2] I have some project on this question.

where $X \in \mathbf{R}^3 \times \mathbf{R}$, $C \in \mathbf{R}^4$ and L is a linear automorphism of ds^2, a *Lorentz transformation*. Let $X = (r, t) \in \mathbf{R}^3 \times \mathbf{R}$, every element of the Lorentz group decompose uniquely in the product of 2 matrices

$$L = \begin{pmatrix} 1_3 & \beta \\ \bar{\beta} & 1 \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & a \end{pmatrix}, \text{ with } \bar{B}B + \beta\bar{\beta} = 1_3 \text{ and } a = \pm\frac{1}{\sqrt{1-\beta^2}}.$$

B is a 3×3 matrix, β is a vector in $\mathbf{R}^3$, the bar over B ou β denotes the transposition operator, and $\beta = \|\beta\|$.

The Poincaré group is also a 10-dimensional Lie group but with 4 connected components. It is customary to reduce the Poincaré group to its identity component. Again,

Principle. *An isolated Einstein relativistic system is a presymplectic manifold* (M, ω) *with a symmetric action of the Poincaré group.*

47. Coadjoint orbits

193. Coadjoint action and orbits. Given a Lie group G, there is a universal model of symplectic manifold. Consider a left invariant 1-form α and let $\mathcal{G}^*$ be the space of left-invariant 1-forms.

$$\mathcal{G}^* = \{\alpha \in \Omega^1(G) \mid L(g)^*(\alpha) = \alpha\},$$

where $L(g) : g' \mapsto gg'$ is the left-multiplication. By invariance every element of $\mathcal{G}^*$ is uniquely determined by its value at the origin, that makes $\mathcal{G}^*$ a vector space with dimension $\dim(G)$. This space is the space of momenta of the group G, it is also interpreted as the dual of the Lie algebra, but we do not need that.

Now, the group G acts by conjugation on itself and by coadjoint action on $\mathcal{G}^*$:

$$\mathrm{Ad}(g) : g' \mapsto gg'g^{-1}$$

and for all $g \in G$:

$$\mathrm{Ad}_*(g) : \mathcal{G}^* \to \mathcal{G}^* \quad \text{with} \quad \mathrm{Ad}(g)_*(\alpha) = \mathrm{Ad}(g^{-1})^*(\alpha).$$

We define the *coadjoint orbit* of $\alpha \in \mathcal{G}^*$ by

$$\mathcal{O}_\alpha = \{\mathrm{Ad}_*(g)(\alpha) \mid g \in G\},$$

and we equip $\mathcal{O}_\alpha$ with the quotient diffeology

$$\mathcal{O}_\alpha \simeq \mathrm{G}/\mathrm{St}(\alpha),$$

where $\mathrm{St}(\alpha)$ is the stabilizer of α

$$\mathrm{St}(\alpha) = \{g \in \mathrm{G} \mid \mathrm{Ad}_*(g)(\alpha) = \alpha\}.$$

Theorem. *The exterior derivative $d\alpha$ is a presymplectic form on G that have the orbits of $\mathrm{St}(\alpha)$ as characteristics. It follows that $\mathcal{O}_\alpha$ has a natural structure of symplectic manifold. That is, there exists a symplectic form ω on $\mathcal{O}_\alpha$ such that*

$$\mathrm{class}^*(\omega) = d\alpha, \quad \textit{with} \quad \mathrm{class}\colon \mathrm{G} \mapsto \mathrm{G}/\mathrm{St}(\alpha).$$

194. Elementary systems or particles. The Kirillov-Kostant-Souriau theorem on classification of transitive symplectic manifolds leads to interpret coadjoint orbits of the groups of symmetries of the mechanics as *elementary particles*. Thus, in Galilean Mechanics, elementary particles will be coadjoint orbits of the Galilean group. In Einstein Relativity they will be coadjoint orbits of the Poincaré group.

Note that in Galilean Mechanics, a large class of orbits have the type

$$\mathcal{O} = \mathbf{R}^6 \times \mathrm{S}^2,$$

with the symplectic form

$$\omega_{m,s} = m\,\mathrm{Can} \oplus s\mathrm{Surf},$$

where Can is the canonical symplectic form on $\mathbf{R}^6$ and Surf the surface element on S^2. The S^2 part represents the *classical spin* component of the particle, s is the spin and m the mass of the elementary particle.

48. The classic moment map

The impact of symmetries in symplectic mechanics are subsumed in a special map called the *moment map*. It was introduced by Souriau in the 60's and published in [**Sou70**].

195. The classic moment map. Let (M, ω) be a presymplectic manifold. Let G be a Lie group with an action by symmetries on M, that is, a smooth morphism

$$G \ni g \mapsto g_M \in \mathrm{Diff}(X, \omega).$$

We say that G is a group of symmetries. Let G be the Lie algebra, that is the space of smooth homomorphisms

$$\mathcal{G} = \mathrm{Hom}^\infty(\mathbf{R}, G).$$

It is identified with the space of invariant vector fields

$$Z_G(g) = \frac{d}{dt} h(t) \cdot g \bigg|_{t=0}, \quad \text{with} \quad Z = Z_G(1_G) = \frac{dh(t)}{dt}\bigg|_{t=0}$$

Every element of the Lie algebra Z defines a vector field Z_M on M, called the *infinitesimal action* of G on M

$$Z_M(x) = \frac{d}{dt} h(t)_M(x) \bigg|_{t=0},$$

Applying the Cartan formula

$$\mathcal{L}_\xi(\varepsilon) = [d\varepsilon](\xi) + d[\varepsilon(\xi)],$$

where $\mathcal{L}_\xi$ denotes the Lie derivative by ξ, for any differential form ε and any vector field ξ, to ω and Z_M, we get

$$d[\omega(Z_M)] = 0,$$

where $\omega(Z_M)$ is the contraction of ω by the vector field Z_M.

Theorem-Definition. *We say that the action of* G *on* M *is* Hamiltonian *if* $\omega(Z_M)$*, which is closed, is exact. Then, there exists a map*

$$\mu\colon M \to \mathcal{G}^* \quad \textit{such that} \quad \omega(Z_M) = d[x \mapsto mu(x) \cdot Z].$$

This map is called the moment map, *it is defined up to a constant, the manifold* M *is assumed to be connected.*

196. The Noether-Souriau theorem. Let (M, ω) be a presymplectic manifold. Let G a Lie group equipped with an Hamiltonian action on M. Then the moment map $\mu\colon M \to \mathcal{G}^*$ is constant on the characteristics. If the quotient $\mathcal{M} = M/\ker(\omega)$ is a manifold, then the moment map descends on $\mathcal{M}$. The moment map μ represents the invariants associated with the symmetries represented by G.

197. The Souriau cocycle. For a presymplectic manifold (M, ω) with a Hamiltonian action of G, with moment μ, we can check the variance of μ according to the action of G on M and the coadjoint action of G on $\mathcal{G}^*$. That gives the following theorem:

Theorem. (Souriau) *The lack of equivariance of the moment map μ is a cocycle of the group G with values in $\mathcal{G}^*$, twisted by the coadjoint action:*

$$\theta(g) = \mu(g_M(x)) - \mathrm{Ad}^*(g)(\mu(x)).$$

The choice of another moment map change θ by a coboundary. Thus, the class is well defined and depends only on the form ω and the action of G.

$$\mathrm{class}(\theta) \in H^1(G, \mathcal{G}^*).$$

I call this cocycle the *Souriau cocycle.*

We recall that these cocycles are defined as map $\theta \colon G \to \mathcal{G}^*$ such that

$$\theta(gg') = \mathrm{Ad}_*(g)(\theta(g')) + \theta(g).$$

A coboundary is a map

$$\Delta(\varepsilon)(g) = \mathrm{Ad}_*(g)(\varepsilon) - c,$$

for all $\varepsilon \in \mathcal{G}^*$.

A major consequence of the moment map and its lack of equivariance is the general theorem of barycentric decomposition.

198. The Barycentric decomposition theorem (Souriau). Let (M, ω) be an isolated dynamical system, that is, a symplectic manifold with an Hamiltonian action of the Galilean group. First of all let us recall that, considering its cohomology:

Theorem. (Bargmann) *The cohomology of the Galilean group is 1-dimensional. Therefore the Souria cocycle θ is equivalent to $m\theta_0$, where θ_0 is a chosen unit.*

The number m is interpreted as the total mass of the system in the unit θ_0.

Theorem. *For an isolated dynamical system* (M, ω), *if the total mass of the system is not zero then the manifold* M *is a product*

$$M = \mathbf{R}^6 \times M_0,$$

where $\mathbf{R}^6$ *represents the motions of the center of gravity and* M_0 *the motions around the center of gravity. The group* $SO(3) \times \mathbf{R}$ *continues to act on* M_0.

199. The Kostant-Kirillov-Souriau theorem. The following theorem is due under different formulations to Kostant, Kirillov and Souriau. I give here the Souriau's formulation.

Theorem. *Let* (M, ω) *be a symplectic manifold, transitive under a Hamiltonian action of a Lie group* G. *Then, the moment map* $\mu \colon M \to \mathcal{G}^*$ *is a covering onto a coadjoint orbit, may be affine.*

Let θ be the Souriau cocycle of the system, it modifies the coadjoint action by adjunction to the standard linear coadjoint ation

$$\mathrm{Ad}_*^{\theta}(g) \colon \varepsilon \mapsto \mathrm{Ad}_*(g)(\varepsilon) + \theta(g).$$

This action is called an *affine coadjoint action*.

200. Exemple: The cylinder and $\mathrm{SL}(2, \mathbf{R})$**.** The group $\mathrm{SL}(2, \mathbf{R})$ acts transitively on the cylinder $\mathbf{R}^2 - \{0\}$, preserving the symplectic form $\mathrm{Surf} = dx \wedge dy$. And the moment map is given by

$$\mu(z)(F_\sigma) = \frac{1}{2}\mathrm{Surf}(z, \sigma z) \times dt,$$

where $z = (x, y) \in \mathbf{R}^2 - \{0\}$, and

$$F_\sigma = \left[s \mapsto e^{s\sigma}\right]$$

is the one-parameter group defined by

$$\sigma \in \mathfrak{sl}(2, \mathbf{R}),$$

the Lie algebra of $\mathrm{SL}(2, \mathbf{R})$, vector space of real 2×2 traceless matrices:

$$\mathfrak{sl}(2, \mathbf{R}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbf{R} \right\}$$

We have clearly $\mu(z) = \mu(-z)$.

201. Dynamic variable and Poisson bracket. Let (M, ω) a symplectic manifold. Every compactly supported real function $[x \mapsto u]$ defines a 1-parameter group of symplectomorphisms. The *symplectic gradient* is uniquely defined by the equation:

$$\omega\Big(\operatorname{grad}_\omega(u), \delta x\Big) = -\delta u \quad \text{with} \quad \delta u = \frac{\partial u}{\partial x}(\delta x).$$

Since u is compactly supported, $\operatorname{grad}_\omega(u)$ is compactly supported too, and then integrable. The 1-parameter group generated by $\operatorname{grad}_\omega(u)$ is denoted by:

$$s \mapsto e^{s\operatorname{grad}_\omega(u)}.$$

We have then,

$$\mathcal{L}_\xi(\omega) = 0, \quad \text{with} \quad \xi = \operatorname{grad}_\omega(u);$$

and

$$\left(e^{s\operatorname{grad}_\omega(u)}\right)^*(\omega) = \omega.$$

Note 1. The function $[x \mapsto u]$ is the moment map associated with the symmetry $\left(e^{s\operatorname{grad}_\omega(u)}\right)_{s\in\mathbf{R}}$.

Note 2. The *Poisson bracket* of two dynamical variables $[x \mapsto u]$ and $[x \mapsto v]$ is defined and denoted by:

$$\{u, v\} = \omega\Big(\operatorname{grad}_\omega(u), \operatorname{grad}_\omega(v)\Big).$$

It satisfies the identity

$$\operatorname{grad}_\omega\big(\{u, v\}\big) = \Big[\operatorname{grad}_\omega(u), \operatorname{grad}_\omega(v)\Big],$$

where the right is the bracket of vector fields. The Poisson bracket is a morphism of algebras.

49. Geometric quantization

The program of geometric quantization try to answer the Dirac program of quantization. It consist, for any symplectic manifold (M, ω), to find a Hilbert space $\mathcal{H}$ and a morphism from the algebra of the real functions, for the Poisson bracket, the albegra of unitary operators:

$$u \mapsto \hat{u} \quad \text{such that} \quad \widehat{\{u, v\}} = [\hat{u}, \hat{v}],$$

and

$$\hat{1} = \mathbf{1}_{\mathcal{H}},$$

where here 1 denotes the constant function $[x \mapsto 1]$.

The expected solution would have been

$$\mathcal{H} = L^2(M) \quad \text{and} \quad \hat{u}(\phi) = L_\xi(\phi)$$

with

$$\xi = \mathrm{grad}_\omega(u), \quad \text{and for all} \quad \phi \in \mathcal{H}.$$

Unfortunately, in that case

$$\hat{1} = 0.$$

The first step in the direction of a solution to this problem is given by the *prequentization construction*.

202. Prequantization. Consider a symplectic manifold (M, ω). Let P_ω be its *group of periods* :

$$P_\omega = \left\{ \int_\sigma \omega \mid \sigma \in H_2(M, \mathbf{Z}) \right\}$$

Then, according to [**PIZ95**] :

Theorem. *There exists always a principal fiber bundle* Y *over* M*, with group* $T_\omega = \mathbf{R}/P_\omega$*, and equipped with a connection form* λ *of curvature* ω*. That is,*

$$d\lambda = \pi^*(\omega),$$

where $\pi \colon Y \to M$*.*

Look in [**TB**, § 8.37] for the definition of a connection form on a principal bundle with group a diffeological torus $\mathbf{R}/P$, where P is a strict subgroup. Note that there maybe more than 1 such *integration bundles*, the classification is given in the paper cited above.

Definition. [**Sou70**] *A symplectic manifold* (M, ω) *is* quantizable *if its group of periods is* $P_\omega = \hbar\mathbf{Z}$*.*

In this case the integration bundle Y is a manifold, a S^1-principal bundle. There exists a unique *fundamental vector field* τ on Y such that:

$$\lambda(\tau) = 1 \quad \text{and} \quad d\lambda(\tau) = 0.$$

Consider a dynamical variable $x \mapsto u$, lift it by π, $[y \mapsto u]$. Then, define the quantized lifting $\widehat{\mathrm{grad}}_\omega(u)$ of $\mathrm{grad}_\omega(u)$ on Y by

$$\widehat{\mathrm{grad}}_\omega(u) = u \times \tau + \eta_u,$$

where η_u is defined by

$$d\lambda(\eta_u) = 0 \quad \text{and} \quad \pi_*(\eta_u) = \mathrm{grad}_\omega(u).$$

Now, let $\mathcal{H}$ be the set of L^2 complex valued function on Y satisfying the equivariant condition

$$\phi(z \cdot y) = z\phi(y),$$

where $z \cdot y$ is the action of $z \in \mathrm{U}(1)$ on $y \in \mathrm{Y}$. Let $y \mapsto u$ the pullback on Y of $x \mapsto u$. Define

$$\hat{u}(\phi) = \frac{\partial \phi}{\partial y}\big(\widehat{\mathrm{grad}}_\omega(u)\big).$$

We check esealy now that

$$\hat{1} = 1 \times \tau + 0 \Rightarrow \hat{1}(\phi) = \frac{\partial \phi}{\partial y}\big(\tau(y)\big) = \phi.$$

The other part of the condition $\widehat{\{u, v\}} = [\hat{u}, \hat{v}]$ is still satisfied.

That construction is called the *prequantization*.

203. The Dirac program, almost. With the prequantization we have a good quandiate which would be perfect if it satisfied the Dirac conditions. Indeed a wave function ϕ in prequatization is, up to a phase, a function on the symplectic manifold, that is, $2n$ variables. There are n variables too much. In the simplest case the symplectic manifold is $\mathbf{R}^n \times \mathbf{R}^n$, space of pairs (q, p), position q and momentum p. The wave function is a function only on the poisition or momentum, or a mix of both, but no more than n variables.

So, to resolve that problem, geometric quantization had proposed to use a *polarization*, that is a projection

$$\pi \colon \mathrm{X} \to \mathrm{Q}$$

which is *a priori* a fibration, where the leaves are Lagrangian subspaces, that is n-dimensional subspaces where ω vanishes.

$$\omega \upharpoonright \pi^{-1}(q) = 0 \quad \text{for all } q \in Q.$$

The wave function would be, more or less, a function on Q. But to represent the 1-parameter group associated with a dynamical variable $x \mapsto u$ the polarization needs to be invariant by that group. Unfortuantely, that almost never happens.

For example in the simplest example of the 2-dimensional *harmonic oscillator*, where the space is just $\mathbf{R} \times \mathbf{R}$ and the group willing to be represented SO(2). The ordinary polarization $(q, p) \mapsto q$, or $(q, p) \mapsto p$, is not invariant by the matrices

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} q \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta)q \\ \sin(\theta)q \end{pmatrix}.$$

The method of geometric quantization is front of a deep problem until now without clear solution.

The case of the harmonic oscillator has been solved by the *pairing method*, which is too long to explain in that kind of short survey notes, but has been satisfactory solved, see for example [**Sou75**].

In conclusion, the problem has been posed by Dirac, and been solved for some part by the geometric quantization program, but remains largely open until today.

50. Symplectic diffeology

In the last decades many examples of wannabe symplectic spaces in infinite dimension came from physics. Also physicists were more and more interested in symplectic constructions involving singularities. These two directions are not covered by traditional symplectic geometry and need a new framework. What was usually done is that to each new example or new situation one creates a specific framework specially adapted to that situation. We call that the *heuristic approach*.

On the other hand, diffeology has all the qualities necessary to fill this program to extend symplectic geometry into the direction of infinite dimensional spaces and singular situations. This is what we shall discuss now. But we need to think and redefine some fundamental definitions to frame correctly the field of symplectic diffeology. Especially, what does mean to be symplectic, to start with.

204. The Darboux condition in diffeology. One most striking property symplectic manifolds share is the Darboux theorem. That is, every symplectic manifold (M, ω) is locally equivalent to $\mathbf{R}^{2n}$ equipped with the standard symplectic structure. That can be rephrased as follow:

205. Theorem (Darboux). *Let* M *be a manifold and* ω *be a symplectic form on* M*. Then, the pseudogroup of local automorphisms* $\mathrm{Diff}_{\mathrm{loc}}(M, \omega)$ *is transitive.*

Of course, that does not say nothing about the local structure itself. We shall come back on that question, but that does not implies that the form ω is non degenerate. It just implies that ω is presymplectic.

Now, since we deal with various properties of closed 2-forms we introduce the general definition:

Definition 1. *We call* parasymplectic form *on a diffeological space* X*, any closed* 2*-form* ω *on* X*.*

Now we introduce:

Definition 2. *We call* presymplectic form *on a diffeological space* X*, any parasymplectic form* ω *on* X *such that the pseudogroup of automorphisms* $\mathrm{Diff}_{\mathrm{loc}}(X, \omega)$ *is transitive.*

If that condition is sufficient to determine the germ of ω at each point as being the germ of $(\mathbf{R}^{2n} \times \mathbf{R}^{k}, \mathrm{Can})$, it is not necessarilly the case in diffeology. That leads to a specific invariant in symplectic diffeology

Definition 3. *The* type *of a presymplectic structure in diffeology will be defined as some representant of equivalent* $\mathrm{germ}_x(\omega)$*, where* (X, ω) *run over presymplectic spaces and* $x \in X$*.*

For example, $(\mathbf{R}^{2n} \times \mathbf{R}^k, \mathrm{Can})$ is the type of all finite dimension presymplectic manifolds.

The question now is to understand how can we characterize the symplectic forms from the presymplectic ones? For that we need the help of the moment map in diffeology.

206. The moment map in diffeology. Next, we consider the group of *symmetries* (or *automorphisms*) of ω, denoted by $\mathrm{Diff}(X, \omega)$. Then, to introduce the *moment map* for any group of symmetries G, we need to clarify some vocabulary and notations:

Definition 1. *A diffeological group is a group that is a diffeological space such that the multiplication and the inversion are smooth.*

Definition 2. *A* momentum *(Plural* momenta*) of a diffeological group* G *is any left-invariant* 1*-form on* G*. We denote by* $\mathcal{G}^*$ *the space of* momenta *:*

$$\mathcal{G}^* = \{\varepsilon \in \Omega^1(G) \mid L(g)^*(\varepsilon) = \varepsilon, \text{ for all } g \in G\}.$$

The set $\mathcal{G}^*$ *is a real vector space. It is also a diffeological vector space for the functional diffeology, but we shall not discuss that point here.*

Next, let (X, ω) be a parasymplectic space and G be a diffeological group.

Definition 3. *A* symmetric action *of* G *on* (X, ω) *is a smooth morphism*

$$g \mapsto g_X \text{ from } G \text{ to } \mathrm{Diff}(X, \omega),$$

where $\mathrm{Diff}(X, \omega)$ *is equipped with the functional diffeology. That is,*

$$\text{for all } g \in G, \quad g_X^*(\omega) = \omega.$$

Now, to grab the essential nature of the moment map, which is a map from X to $\mathcal{G}^*$, we need to understand it in the simplest possible case. That is, when:

(1) ω is exact, $\omega = d\alpha$,
(2) and when α is also invariant by G, $g_X^*(\alpha) = \alpha$.

In these conditions, the moment map is given by

$$\mu\colon X \to \mathcal{G}^* \quad \text{with} \quad \mu(x) = \hat{x}^*(\alpha),$$

where

$$\hat{x}\colon G \to X$$

is the *orbit map*

$$\hat{x}(g) = g_X(x).$$

We check immediately that,

Proposition 1. *Since* α *is invariant by* G, $\hat{x}^*(\alpha)$ *is left invariant by* G, *and therefore*

$$\mu(x) \in \mathcal{G}^*.$$

Actually, as we know that:

Not all closed 2-forms are exact, and even if they are exact, they do not necessarily have an invariant primitive.

We shall see now, how we can generally come to a situation, so close to the simple case above, that, modulo some minor subtleties, we can build a good moment map in all cases.

Let us consider now the general case, with X connected. Let $\mathcal{K}$ be the chain-homotopy operator, defined in [TB, § 6.83] :

$$\mathcal{K}\colon \Omega^k(X) \to \Omega^{k-1}(\text{Paths}(X)) \quad \text{with} \quad \mathcal{K} \circ d + d \circ \mathcal{K} = \hat{1}^* - \hat{0}^*.$$

Then, the differential 1-form Kω, defined on Paths(X), satisfies

$$d[\mathcal{K}\omega] = (\hat{1}^* - \hat{0}^*)(\omega),$$

and $\mathcal{K}\omega$ is invariant by G [TB, § 6.84]. Considering

$$\bar{\omega} = (\hat{1}^* - \hat{0}^*)(\omega)$$

and

$$\bar{\alpha} = \mathcal{K}\omega,$$

we are in the simple case:

$$\bar{\omega} = d\bar{\alpha}$$

and $\bar{\alpha}$ invariant by G. We can apply the construction above and then:

<u>Definition 4.</u> *We define the* paths moment map *by*

$$\Psi\colon \operatorname{Paths}(X) \to \mathcal{G}^* \quad \textit{with} \quad \Psi(\gamma) = \hat{\gamma}^*(\mathcal{K}\omega),$$

where $\hat{\gamma}\colon G \to \operatorname{Paths}(X)$ *is the orbit map* $\hat{\gamma}(g) = g_X \circ \gamma$ *of the path* γ.

The paths moment map is additive with respect to the concatenation,

$$\Psi(\gamma \vee \gamma') = \Psi(\gamma) + \Psi(\gamma'),$$

and it is equivariant by G, which acts by composition on Paths(X), and by coadjoint action on $\mathcal{G}^*$. That is, for all $g, k \in G$ and $\varepsilon \in \mathcal{G}^*$,

$$\operatorname{Ad}(g)\colon k \mapsto gkg^{-1}$$

and

$$\operatorname{Ad}_*(g)\colon \varepsilon \mapsto \operatorname{Ad}(g)_*(\varepsilon) = \operatorname{Ad}(g^{-1})^*(\varepsilon).$$

Then,

<u>Definition 5.</u> *We define the* holonomy *of the action of* G *on* X *as the subgroup*

$$\Gamma = \{\Psi(\ell) \mid \ell \in \operatorname{Loops}(X)\} \subset \mathcal{G}^*.$$

<u>Proposition 2.</u> *The group* Γ *is made of (closed)* Ad_**-invariant momenta. But* $\Psi(\ell)$ *depends only on the homotopy class of* ℓ*, so then* Γ *is a homomorphic image of* $\pi_1(X)$*, more precisely, its abelianized.*

<u>Definition 6.</u> *If* $\Gamma = \{0\}$*, the action of* G *on* (X, ω) *is said to be* Hamiltonian*. The holonomy* Γ *is the obstruction for the action of the group* G *to be Hamiltonian.*

Now, we can push forward the paths moment map on $\mathcal{G}^*/\Gamma$, as suggested by the commutative diagram

$$\begin{array}{ccc}
\operatorname{Paths}(X) & \xrightarrow{\ \Psi\ } & \mathcal{G}^* \\
{\scriptstyle \text{ends}}\downarrow & & \downarrow{\scriptstyle \text{class}} \\
X \times X & \xrightarrow[\ \psi\ ]{} & \mathcal{G}^*/\Gamma
\end{array}$$

and we get then:

Defintion 7. *The* two-points moment map *is defined by:*

$$\psi(x, x') = \mathrm{class}(\Psi(\gamma)) \in \mathcal{G}^*/\Gamma,$$

for any path γ *such that* $\mathrm{ends}(\gamma) = (x, x')$.

Proposition 3. *The additivity of* Ψ *becomes the Chasles's cocycle condition on* ψ *:*

$$\psi(x, x') + \psi(x', x'') = \psi(x, x'').$$

Since the group Γ is invariant by the coadjoint action, the coadjoint action passes to the quotient group $\mathcal{G}^*/\Gamma$, and ψ is a natural *group-valued moment map*, equivariant for this quotient coadjoint action.

Definition 8. *Because* X *is connected, there exists always a map*

$$\mu\colon X \to \mathcal{G}^*/\Gamma \quad \textit{such that} \quad \psi(x, x') = \mu(x') - \mu(x).$$

The solutions of this equation are given by

$$\mu(x) = \psi(x_0, x) + c,$$

where x_0 *is a chosen point in* X *and* c *is a constant. These are the* one-point moment maps.

But these moment maps μ are a priori no longer equivariant. Their variance introduces a 1-cocycle θ of G with values in $\mathcal{G}^*/\Gamma$.

Definition 9. *Let* $x_0 \in X$ *and* $c \in \mathcal{G}^*/\Gamma$. *We define:*

$$\theta(g) = \psi(x_0, g(x_0)) + \Delta c(g),$$

for all $g \in G$, *with*

$$\Delta c(g) = \mathrm{Ad}_*(g)(c) - c$$

Then, $\theta(g)$ *is a 1-cocycle of* G *with values in* $\mathcal{G}^*/\Gamma$, *twisted by* Ad_*, *and* Δc *is a coboundary. Moreover,*

$$\mu(g(x)) = \mathrm{Ad}_*(g)(\mu(x)) + \theta(g),$$

Changing the base point x_0 *and the constant* c *in* μ *changes the cocycle* θ *into a equivalent cocycle.*

The cocycle θ capture the lack of invariance of the moment μ, we called it the <u>*Souriau's cocycle*</u> *since it is a generalization of the manifold case. The cohomology class*

$$\sigma = \mathrm{class}(\theta) \in \mathrm{H}^1(\mathrm{G}, \mathcal{G}^*/\Gamma)$$

is uniquely defined by the action of G *on* X*. We say that the action of* G *on* (X, ω) *is* exact *when* $\sigma = 0$*, that is, when the cocycle* θ *is a coboundary.*

Next, defining

$$\mathrm{Ad}_*^\theta(g) \colon \nu \mapsto \mathrm{Ad}_*(g)(\nu) + \theta(g),$$

then

$$\mathrm{Ad}_*^\theta(gg') = \mathrm{Ad}_*^\theta(g) \circ \mathrm{Ad}_*^\theta(g').$$

The cocycle property of θ, that is,

$$\theta(gg') = \mathrm{Ad}_*(g)(\theta(g')) + \theta(g),$$

makes Ad_*^θ an action of G on the group $\mathcal{G}^*/\Gamma$. This action is called the *affine action*.

<u>Proposition 4.</u> *For the affine action, the moment map* μ *is equivariant:*

$$\mu(g(x)) = \mathrm{Ad}_*^\theta(g)(\mu(x)).$$

This construction extends to the category {Diffeology}, the moment map for manifolds introduced by Souriau in [**Sou70**]. When X is a manifold and the action of G is Hamiltonian, they are the standard moment maps he defined there.

The remarkable and very important point is this: none of the constructions brought up above involves differential equations, and there is no need for considering a potential Lie algebra either.

The momenta appear as invariant 1-forms on the group, naturally, without intermediaries, and the moment map as a map in the space of momenta.

Note that the group of automorphisms $\mathrm{G}_\omega = \mathrm{Diff}(\mathrm{X}, \omega)$ is a legitimate diffeological group. The above constructions apply and give rise to universal objects:

- *universal momenta* $\mathcal{G}^*_\omega$,
- *universal path moment map* Ψ_ω,
- *universal holonomy* Γ_ω,
- *universal two-points moment map* ψ_ω,
- *universal moment maps* μ_ω,
- *universal Souriau's cocycles* θ_ω, and *cohomology class* σ_ω.

Note. The universal cohomology class σ_ω is a *parasymplectif invariant* depeding only on (M, ω).

A *parasymplectic action* of a diffeological group G is a smooth morphism $h\colon \mathrm{G} \to \mathrm{G}_\omega$, and the objects, associated with G, introduced by the above moment maps constructions, are naturally subordinate to their universal counterparts.

Many examples can be found in [**TB**, Sections 9.27 – 9.34]

207. Example: The moment of imprimitivity. Consider the cotangent space $\mathrm{T}^*\mathrm{M}$ of a manifold M, equipped with the standard symplectic form

$$\omega = d\lambda,$$

where λ is the Liouville form:

$$\lambda_{(x,a)}\left(\frac{d(x,a)}{ds}\right) = a\left(\frac{dx}{ds}\right).$$

Let G be the Abelian group

$$\mathrm{G} = \mathcal{C}^\infty(\mathrm{M}, \mathbf{R}).$$

Consider the action of G on $\mathrm{T}^*\mathrm{M}$ defined by

$$f\colon (x,a) \mapsto (x, a - df_x),$$

where $x \in \mathrm{M}$, $a \in \mathrm{T}^*_x\mathrm{M}$, and df_x is the differential of f at the point x. Then, the moment map is given by

$$\mu\colon (x,a) \mapsto d[f \mapsto f(x)] = d[\delta_x],$$

where δ denotes here the Dirac distribution $\delta_x(f) = f(x)$.[3]

[3]This example is inspired from a heuristic in [**Zie96**].

We see that in this case, the moment map identifies with a function with values distributions but still has the definite formal statute of a map into the space of momenta of the group of symmetries.

Moreover, this action is Hamiltonian and exact. This example generalizes to cotangent of diffeological spaces, see [TB, Exercise 147].

☞ Proof. Since δ_x is a smooth function on $\mathcal{C}^\infty(M, \mathbf{R})$, its differential is a 1-form.[4] Let us check that this 1-form is invariant: Let $h \in \mathcal{C}^\infty(M, \mathbf{R})$, $L(h)^*(\mu(x)) = L(h)^*(d[\delta_x]) = d[L(h)^*(\delta_x)] = d[\delta_x \circ L(h)]$, but $\delta_x \circ L(h) \colon f \mapsto \delta_x(f + h) = f(x) + h(x)$. Then, $d[\delta_x \circ L(h)] = d[f \mapsto f(x) + h(x)] = d[f \mapsto f(x)]$. Therefore, $L(h)^*(\mu(x)) = \mu(x)$. ▶

208. Symplectic manifolds are coadjoint orbits. Because symplectic forms of manifolds have no local invariants, as we know thanks to Darboux's theorem, they have a huge group of automorphisms. This group is big enough to be transitive [Boo69], so that we will be able to identify the symplectic manifold with its image by the universal moment map. Then, by equivariance, it will give a coadjoint orbit (affine or not) of its group of symmetries. In other words, coadjoint orbits are the universal models of symplectic manifolds.

Precisely, let M be a connected Hausdorff manifold, and let ω be a closed 2-form on M. Let $G_\omega = \mathrm{Diff}(M, \omega)$ be its group of symmetries and $\mathcal{G}^*_\omega$ its space of momenta. Let Γ_ω be the holonomy, and μ_ω be a universal moment map with values in $\mathcal{G}^*_\omega/\Gamma_\omega$. We have, then, the following:

Theorem 1. (P.I-Z) *The form ω is symplectic, that is non-degenerate, if and only if:*

1. *the group G_ω is (locally) transitive on M;*
2. *the universal moment map $\mu_\omega \colon M \to \mathcal{G}^*_\omega/\Gamma_\omega$ is injective.*

This theorem is proved in [TB, § 9.23], but let us make some comments on the key elements.

[4]This deserves to be emphasized: the exterior derivative of the Dirac distribution exists and is a differential 1-form on the group of real functions.

Remark Consider the closed 2-form $\omega = (x^2 + y^2)\,dx \wedge dy$; one can show that it has an injective universal moment map μ_ω. But its group G_ω is not transitive, since ω is degenerate in $(0,0)$, and only at that point. Thus, the transitivity of G_ω is necessary.

The case homogeneous presymplectic manifolds is interesting for what it suggests:

Theorem 2. (P.I-Z) *Let* (M, ω) *be a presymplectic manifold, homogeneous under its group of automorphisms. Then, the characteristcis of* ω *are the preimages of the universal moment map* μ_ω*.*

Proof. Let us give some hints about the sequel of the proof. Assume ω is symplectic. Let $m_0, m_1 \in M$ and p be a path connecting these points. For all $f \in \mathcal{C}^\infty(M, \mathbf{R})$ with compact support, let

$$F \colon t \mapsto e^{t\,\mathrm{grad}_\omega(f)}$$

be the exponential of the symplectic gradient of the f. Then, F is a 1-parameter group of automorphisms, and its value on $\Psi_\omega(p)$ is:

$$\Psi_\omega(p)(F) = [f(m_1) - f(m_0)] \times dt.$$

Now, if $\mu_\omega(m_0) = \mu_\omega(m_1)$, then there exists a loop ℓ in M such that $\Psi_\omega(p) = \Psi_\omega(\ell)$. Applied to the 1-plot F, we deduce $f(m_1) = f(m_0)$ for all f. Therefore $m_0 = m_1$, and μ_ω is injective.

Conversely, let us assume that G_ω is transitive, and μ_ω is injective. By transitivity, the rank of ω is constant. Now, let us assume that ω is degenerate, that is, $\dim(\ker(\omega)) > 0$. Since the distribution $\ker(\omega)$ is integrable, given two different points m_0 and m_1 in a characteristic, there exists a path p connecting these two points and drawn entirely in the characteristic, that is, such that $dp(t)/dt \in \ker(\omega)$ for all t. But that implies $\Psi_\omega(p) = 0$ [TB, § 9.20]. Hence, $\mu_\omega(m_0) = \mu_\omega(m_1)$. But we assumed μ_ω is injective. Thus, ω is nondegenerate, that is, symplectic. ►

209. Symplectic diffeological spaces. Thanks to the two previous theorems I proposed a definition of symplectic diffeological space:

Definition. *A parasymplectic form* ω *on a diffeological space* X *will be sait symplectic if:*

(1) *The local automorphisms* $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{X}, \omega)$ *are transitive.*
(2) *The universal moment map* μ_ω*is a covering onto its image.*

The first condition means that ω is presymplectic, we'll call it the Darboux condition. The second condition mimic the situation of manifolds, but we can hardly ask the universal moment map to be injective, we do not know enough. The weaker condition of being a covering would be probably sufficient to insure the symplectic nature of the presymplectic form. It is possible that we can weaken the second condition by considering an equivalent of the universal moment map for the pseudogroup of local automorphisms. But that remain to be investigated.

Note. There is a conflict between the definition I gave above and what is usually regarded as "symplectic orbifold". For example, the symplectic form $\omega = dx \wedge dy$ descends on the quotient space $\mathcal{Q}_m = \mathbf{C}/\mathcal{U}_m$, where $\mathcal{U}_m$ is the group of m-th roots of unity. And this space is regarded as "symplectic" in the literature. However, it does not fit the definition above because the pseudogroup of automorphisms fixes the origin $0 \in \mathcal{Q}_m$, even though the universal moment map is injective.

To relsolve this dilemma about the use of the word "symplectic" outside the realm of differentiable manifolds, I suggest to call this kind of parasymplectic spaces, like the cone orbifold, symplectically generated, what they are. This opens a lot of questions by comparing the two kinds of parasymplectic spaces.

Example: The infinite projective space:

$$\mathrm{CP}^\infty = \mathrm{S}^\infty/\mathrm{S}^1,$$

where $\mathrm{S}^\infty \subset \ell^2$ is the set of infinite complex sequences of norm 1, is a symplectic diffeological space. It is actually a coadjoint orbit of the diffeological group $\mathrm{U}(\mathcal{H})$.

Diffeology and Non-Commutative Geometry

In this lecture we show how we can build a bridge between some diffeological spaces and noncommutative geometry, such that diffeomorphic spaces give Morita equivalent $\mathbf{C}^*$-algebras. These spaces are orbifolds, generalized by quasifolds, regarded as diffeological spaces.

The basic ideas, at the source of the relationship between diffeology noncommutative geometry, have been introduced in a first paper [**IZL18**] of the following two papers, and its results extended in the second one [**IZP21**]:

- "Noncommutative geometry and diffeology: The case of orbifolds",
- "Quasifolds, diffeology and noncommutative geometry".

We have seen that the concept of orbifold has been introduced by Ishiro Stake [**Sat56, Sat57**] as V-manifolds, and renamed orbifolds by Thurston [**Thu78**]. On the other hand, its generalization to quasifolds was proposed by Elisa Prato [**EP01**].

In the paper "Orbifolds as diffeologies" [**IKZ10**] we include the orbifolds in the category {Diffeology}. In the second paper we give a diffeological definition of "quasifolds" that fits correctly the original definition. Thus, quasifolds are also included in the category {Diffeology}. By this inclusive diffeological approach it is obvious that the Elisa Prato quasifolds are a generalization of orbifolds, which is not necessarily obvious with the specific definitions. We have then this particular series of full subcategories:

$$\{\text{Manifolds}\} \preceq \{\text{Orbifolds}\} \preceq \{\text{Quasifolds}\} \preceq \{\text{Diffeology}\}$$

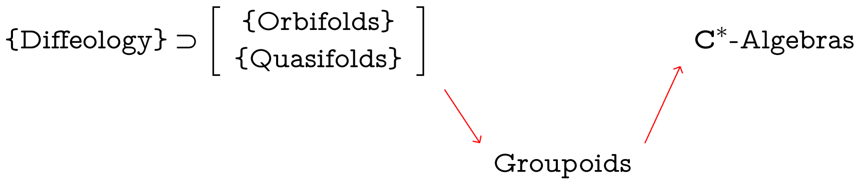
Then, we associate a $\mathbf{C}^*$-algebra to the orbifold, or quasifold, such that diffeomorphic spaces give Morita equivalent $\mathbf{C}^*$-algebra, which is the minimum required.

I insist on the fact that the process of associating a $\mathbf{C}^*$-algebra to these categories of spaces is not tautological as it can be with a direct algebraic approach which contains already in the definition of the category (based on groupoids or stacks), the particular property that equivalent structures (groupoids or stacks) give Morita equivalent $\mathbf{C}^*$-algebras. In our case, we start here a floor below, if I may say so, with the geometry of the space, that is, its diffeology.

The outline of the construction is as follows:

(1) Definition of orbifolds and quasifolds in diffeology.
(2) Introduction of charts and atlases that define the structure.
(3) Associate with every atlas a strict generating family and its nebula.
(4) Associate a groupoid over the space with the nebula of each atlas, that captures the local structure point by point.
(5) Prove that two different atlases give to equivalent groupoids, in the algebraic sense, which is the minimum required: the groupoid [its class] is a diffeological invariant of the space.
(6) Prove that these groupoids are etale and Hausdorff.
(7) Associate a $*$-algebra, and a $\mathbf{C}^*$-algebra by completion, to each of these groupoids, according to Jean Renault's construction.
(8) Use a central theorem from Muhly-Renault-Willian, proving that two different atlases give Morita equivalent $\mathbf{C}^*$-algebras, which is the minimum expected.
(9) And finally illustration by two examples: the $\mathbf{C}^*$-algebra associated with the irrational torus T_α, and the $\mathbf{C}^*$-algebra associated with the quotient $\mathbf{R}/\mathbf{Q}$.

In the two cases, the idea is to associate a structure groupoid to these objects and then, by a now standard procedure, associate a $\mathbf{C}^*$-algebra.



I would like to finish this preamble by recalling that the development of diffeology, starting in 1983 with the example of the irrational torus [DI83], was deeply motivated by the emergence of noncommutative geometry dealing with quasiperiodic potentials in quantum mechanics. It was time to close the loop, at least for now.

51. Orbifolds and quasifolds again

210. The orbifolds. Let us recall that an orbifold is a diffeological space that is locally diffeomorphic to some quotient $\mathbf{R}^n/\Gamma$, at each point, where Γ is a finite subgroup of $\mathrm{GL}(n,\mathbf{R})$. The group Γ may change from point to point. See lecture "Modeling: Manifolds, orbifolds and quasifolds".

Example 1. The quotient space $\mathcal{Q}_m = \mathbf{C}/\mathcal{U}_m$, with the group of roots of unity $\mathcal{U}_m = \{\exp(2i\pi k/m) \mid k = 1 \dots m\}$, is a *cone-orbifold*.

Example 2. The product $[\mathbf{R}/\{\pm 1\}]^n$ is a *corner-orbifold*.

Example 3. We have seen the *waterdrop* represented in Figure 27. We recall that it is the sphere S^2, equipped with a specific diffeology described previously.

211. The quasifolds. I recall that a quasifold is a diffeological space that is locally diffeomorphic to $\mathbf{R}^n/\Gamma$, where Γ is a countable subgroup of the affine group $\mathrm{Aff}(\mathbf{R}^n)$, $x \mapsto \mathrm{A}x + \mathrm{B}$, with $\mathrm{A} \in \mathrm{GL}(n,\mathbf{R})$ and $\mathrm{B} \in \mathbf{R}^n$. Diffeological quasifolds are a generalization of orbifolds.

Example 1. The first example of quasifold is the *irrational torus*, the first special diffeological space studied for itself in 1983 [DI83], which is at the source of the development of diffeology:

$$T_\alpha = T^2/\mathcal{S}_\alpha \simeq \mathbf{R}/(\mathbf{Z}+\alpha\mathbf{Z}),$$

where $\alpha \in \mathbf{R}-\mathbf{Q}$, $\mathcal{S}_\alpha \subset T^2$ is the projecttion of the line $y = \alpha x$, and $T^2 = [\mathbf{R}/\mathbf{Z}]^2$.

Example 2. The second example $\mathfrak{G}$ (for Geodesics) is inspired by the first one. The lines of slope α are the geodesic trajectories on the torus T^2 of slope α. The set of all geodesic trajectories of the torus T^2 are bundled over S^1, they are the projections on T^2 of all the affine lines in $\mathbf{R}^2$ directed by a unit vector $u \in S^1$. Over the vector u we have $\mathfrak{G}_u$, the torus T_u which is rational or irrational depending if the line $\mathbf{R}u$ cut or not the lattice $\mathbf{Z}^2$ elsewhere than in 0.

The set $\mathfrak{G}$ of the geodesic trajectories of the torus T^2 is the quotient of the space of geodesic trajectories of the plane $\mathbf{R}^2$ by the action of $\mathbf{Z}^2$. The space of geodesic trajectories of the plane is equivalent to the cylinder

$$TS^1 = \{(u,r) \in S^1 \times \mathbf{R}^2 \mid \langle u, r\rangle = 0\}.$$

The mapping

$$(u,r) \mapsto \big(u, \rho = \langle r, Ju\rangle\big) \quad \text{with} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

identifies

$$TS^1 \simeq S^1 \times \mathbf{R}.$$

The action of $\mathbf{Z}^2$ on TS^1 is given by

$$\begin{pmatrix} m \\ n \end{pmatrix} : (u,r) \mapsto \left(u, r + [1 - u\bar{u}] \begin{pmatrix} m \\ n \end{pmatrix}\right).$$

Translated on (u, ρ) that gives:

$$\begin{pmatrix} m \\ n \end{pmatrix} : (u,\rho) \mapsto \left(u, \rho + \left\langle \begin{pmatrix} m \\ n \end{pmatrix}, Ju \right\rangle\right).$$

That is,

$$\begin{pmatrix} m \\ n \end{pmatrix} : (u, \rho) \mapsto (u, \rho + nu_x - mu_y) \quad \text{with} \quad u = \begin{pmatrix} u_x \\ u_y \end{pmatrix}.$$

In other words, $\mathfrak{G}$ is diffeomorphic to the quotient of $\mathbf{R} \times \mathbf{R}$ by the relation

$$(t, \rho) \sim \big(t + \ell, \rho + n\cos(2\pi t) - m\sin(2\pi t)\big) \quad \text{with} \quad \ell, n, m \in \mathbf{Z}.$$

212. Charts, atlases and strict generating families. The definition of charts, atlases and strict generating families for quasifolds are strictly similar for that of orbifolds, but since we did not write them down formally in the chapter "Modeling: Manifolds, orbifolds and quasifolds", we shall take the time to do it here.

Consider a quasifold X, and $x \in X$. Let $\Gamma \subset \mathrm{Aff}(\mathbf{R}^n)$ be a countable sugroup of the affine group of $\mathbf{R}^n$ and ϕ be a local diffeomorphism from $\mathbf{R}^n/\Gamma$ to X, defined on some open subset $U \subset \mathbf{R}^n/\Gamma$, such that $x \in \phi(U)$. The subset U is open for the D-topology, that is in this case, the quotient topology by the projection map $\mathrm{class}\colon \mathbf{R}^n \to \mathbf{R}^n/\Gamma$.

Definition 1. *Any such diffeomorphism is called a* chart. *A set of charts* $\mathcal{A}$, *covering* X, *is called an* atlas.

Let $f\colon U \to X$ be a chart, then U is an open subset of some $\mathbf{R}^n/\Gamma$ for the D-topology. Thus $\tilde{U} = \mathrm{class}^{-1}(U)$ is a Γ-invariant open subset in $\mathbf{R}^n$. Hence, $F = f \circ \mathrm{class}$ is a plot of X. We shall call it the *strict lifting* of f.

Definition 2. *Let* $\mathcal{F}$ *be the set of strict liftings* $F = f \circ \mathrm{class}$, *where* $f\colon U \to X$ *runs over the charts in* $\mathcal{A}$. *Then,* $\mathcal{F}$ *is a generating family of* X. *We shall say that* $\mathcal{F}$ *is the* strict generating family *associated with* $\mathcal{A}$.

52. Structure groupoids

213. Lifting the identity. Let $\mathcal{Q} = \mathbf{R}^n/\Gamma$ where Γ is a countable subgroup of $\mathrm{Aff}(\mathbf{R}^n)$. Consider a local smooth map F from $\mathbf{R}^n$ to itself, such that

$$\mathrm{class} \circ F = \mathrm{class}.$$

In other words, F is a local lifting of the identity on $\mathcal{Q}$. Then,

Theorem. *F is locally equal to some group action*

$$F(r) =_{\mathrm{loc}} \gamma \cdot r = Ar + b,$$

where $\gamma = (A, b) \in \Gamma$, *for some* $A \in GL(\mathbf{R}^n)$ *and* $b \in \mathbf{R}^n$.

☞ Proof. Let us assume first that F is defined on an open ball $\mathcal{B}$. Then, for all r in the ball, there exists a $\gamma \in \Gamma$ such that $F(r) = \gamma \cdot r$. Next, for every $\gamma \in \Gamma$, let

$$F_\gamma : \mathcal{B} \to \mathbf{R}^n \times \mathbf{R}^n \quad \text{with} \quad F_\gamma(r) = (F(r), \gamma \cdot r).$$

Let $\Delta \subset \mathbf{R}^n \times \mathbf{R}^n$ be the diagonal and let us consider

$$\Delta_\gamma = F_\gamma^{-1}(\Delta) = \{r \in \mathcal{B} \mid F(r) = \gamma \cdot r\}.$$

Lemma. There exist at least one $\gamma \in \Gamma$ such that the interior $\mathring{\Delta}_\gamma$ is non-empty, and the union $\mathring{\Delta}_\Gamma = \cup_{\gamma \in \Gamma} \mathring{\Delta}_\gamma$ is an open dense subset of $\mathcal{B}$.

◀ The proof of this lemma is identical to that for orbifolds. ▶

The following is a bit different and deserves to be developed: There exists a subset of Γ, indexed by a family $\mathcal{J}$, for which $\mathcal{O}_i = \mathring{\Delta}_{\gamma_i} \subset \mathcal{B}$ is open and non-empty, $\cup_{i \in \mathcal{J}} \mathcal{O}_i$ is an open dense subset of $\mathcal{B}$, and $F \upharpoonright \mathcal{O}_i : r \mapsto A_i r + b_i$, where $(A_i, b_i) \in \mathrm{Aff}(\mathbf{R}^n)$. Since F is smooth, the first derivative $D(F)$ restricted to $\mathcal{O}_i$ is equal to A_i, and then the second derivative $D^2(F) \upharpoonright \mathcal{O}_i = 0$, for all $i \in \mathcal{J}$. Then, since $D^2(F) = 0$ on an open dense subset of $\mathcal{B}$, $D^2(F) = 0$ on $\mathcal{B}$, that is $D(F)(r) = A$ for all $r \in \mathcal{B}$, with $A \in GL(n, \mathbf{R})$. Now, the map $r \mapsto F(r) - Ar$, defined on $\mathcal{B}$, is smooth. But, restricted on $\mathcal{O}_i$ it is equal to b_i. Its derivative vanishes on the open dense subset $\cup_{i \in \mathcal{J}} \mathcal{O}_i$ and thus vanishes on $\mathcal{B}$. Therefore, $F(r) - Ar = b$ on the whole $\mathcal{B}$, for $b \in \mathbf{R}^n$ and $F(r) = Ar + b$ on $\mathcal{B}$, with $\gamma = (A, b) \in \Gamma$.▶

214. Building the groupoid of a quasifold. The structure groupoid of a quasifold X is built on the same principles as that of orbifolds. Let

$\mathcal{A}$ be an atlas and let $\mathcal{F}$ be the strict generating family over $\mathcal{A}$. Let $\mathcal{N}$ be the nebula of $\mathcal{F}$:

$$\mathcal{N} = \coprod_{\mathrm{F}\in\mathcal{F}} \mathrm{dom}(\mathrm{F}) = \{(\mathrm{F}, r) \mid \mathrm{F} \in \mathcal{F} \text{ and } r \in \mathrm{dom}(\mathrm{F})\}.$$

The *evaluation map* is the natural subduction

$$\mathrm{ev}\colon \mathcal{N} \to \mathrm{X} \quad \text{with} \quad \mathrm{ev}(\mathrm{F}, r) = \mathrm{F}(r).$$

The *structure groupoid* of the quasifold X, associated with the atlas $\mathcal{A}$, is defined as the subgroupoid $\mathbf{G}$ of germs of local diffeomorphisms of $\mathcal{N}$ that project to the identity of X along ev. That is,

$$\begin{cases} \mathrm{Obj}(\mathbf{G}) = \mathcal{N}, \\ \mathrm{Mor}(\mathbf{G}) = \{ \, \mathrm{germ}(\Phi)_\nu \mid \Phi \in \mathrm{Diff_{loc}}(\mathcal{N}) \text{ and } \mathrm{ev} \circ \Phi = \mathrm{ev} \restriction \mathrm{dom}(\Phi)\}. \end{cases}$$

The set $\mathrm{Mor}(\mathbf{G})$ is equipped with the functional diffeology inherited by the full groupoid of germs of local diffeomorphisms.[1] Note that, given $\Phi \in \mathrm{Diff_{loc}}(\mathcal{N})$ and $\nu \in \mathrm{dom}(\Phi)$, there exist always two plots F and F′ in $\mathcal{F}$ such that $\nu = (\mathrm{F}, r)$, with $r \in \mathrm{dom}(\mathrm{F})$, and a local diffeomorphism ϕ of $\mathbf{R}^n$, defined on an open ball centered in r, such that $\mathrm{dom}(\phi) \subset \mathrm{dom}(\mathrm{F})$, $\phi = \Phi \restriction \{\mathrm{F}\}\times\mathrm{dom}(\mathrm{F})$ and $\mathrm{F}' \circ \phi = \mathrm{F} \restriction \mathrm{dom}(\phi)$. That is summarized by the diagram:

$$\begin{array}{ccc} \mathrm{dom}(\mathrm{F}) \supset \mathrm{dom}(\phi) & \xrightarrow{\ \phi\ } & \mathrm{dom}(\mathrm{F}') \\ & {\scriptstyle \mathrm{F}} \searrow \quad \swarrow {\scriptstyle \mathrm{F}'} & \\ & \mathrm{X} & \end{array}$$

Note. According to the theorem in § 213, the local diffeomorphisms, defined on the domain of a generating plot, and lifting the identity of the quasifold, are just the elements of the structure group associated with the plot.

We can legitimately wonder what is the point of involving general germs of local diffeomorphisms, if we merely end up with the structure group we could have began with? The reason is that: *The structure groups connect the points of the nebula that project on a same point*

[1]That is defined precisely in the paper on orbifolds and $\mathbf{C}^*$-algebras [IZL18, § 2 & 3]

of the quasifold, only when they are inside the same domain. They cannot connect the points of the nebula that project on the same point of the quasifold but belonging to different domains, with maybe different structure groups. This is the reason why we cannot avoid the use of germs of local diffeomorphisms in the nebula, to begin with. That situation is illustrated in Figure 33.

215. Lifting local diffeomorphisms. Let $\mathcal{Q} = \mathbf{R}^n/\Gamma$ and $\mathcal{Q}' = \mathbf{R}^{n'}/\Gamma'$, where $\Gamma \subset \mathrm{Aff}(\mathbf{R}^n)$ and $\Gamma' \subset \mathrm{Aff}(\mathbf{R}^{n'})$ are countable subgroups. Then,

Theorem. *Every local smooth lifting $\tilde{f}$ of any local diffeomorphism f of $\mathcal{Q}$ is necessarily a local diffeomorphism. In particular $n = n'$. Moreover, let $x \in \mathrm{dom}(f)$, $x' = f(x)$, $r, r' \in \mathbf{R}^n$ be such that $\mathrm{class}(r) = x$ and $\mathrm{class}(r') = x'$. Then, the local lifting $\tilde{f}$ can be chosen such that $\tilde{f}(r) = r'$.* Note that n is also the diffeological dimension of $\mathbf{R}^n/\Gamma$.

☞ Proof. This theorem is a consequence of the previous theorem on the lifting of the identity ▶

216. Equivalence of structure groupoids. The construction and properties of the structure groupoid associated with a quasifold follow word for word the equivalent construction and properties as in the case of an orbifold.

Proposition. *The fibers of the subduction* $\mathrm{ev}\colon \mathrm{Obj}(\mathbf{G}) \to X$ *are exactly the transitivity-components of* $\mathbf{G}$*. In other words, the space of transitivity components of the groupoid* $\mathbf{G}$ *associated with any atlas of the quasifold* X*, equipped with the quotient diffeology, is the quasifold itself.*

Theorem. *Different atlases of* X *give equivalent structure groupoids. The structure groupoids associated with diffeomorphic quasifolds are equivalent.*

In other words, the equivalence class of the structure groupoids of a quasifold is a diffeological invariant.

53. The C*-algebra

We use the construction of the C*-algebra associated with an arbitrary locally compact groupoid **G**, equipped with a Haar system, introduced and described by Jean Renault in [**JR80**, Part II, § 1]. Note that, for this construction, only the topology of the groupoid is involved, and diffeological groupoids, when regarded as topological groupoids, are equipped with the D-topology.[2]

We will denote by $\mathcal{C}(\mathbf{G})$ the completion of the compactly supported continuous complex functions on $\mathrm{Mor}(\mathbf{G})$, for the uniform norm. And we still consider, as is done for orbifolds, the particular case where the Haar system is given by the *counting measure*. Let f and g be two compactly supported complex functions, the convolution and the involution are defined by

$$f * g(\gamma) = \sum_{\beta \in \mathbf{G}^x} f(\beta \cdot \gamma) g(\beta^{-1}) \quad \text{and} \quad f^*(\gamma) = f(\gamma^{-1})^*.$$

The sums involved are supposed to converge. Here, $\gamma \in \mathrm{Mor}(\mathbf{G})$, $x = \mathrm{src}(\gamma)$ and $\mathbf{G}^x = \mathrm{trg}^{-1}(x)$ is the subset of arrows with target x. The star in z^* denotes the conjugate of the complex number z. By definition, the vector space $\mathcal{C}(\mathbf{G})$, equipped with these two operations, is the C*-algebra associated with the groupoid **G**.

217. The structure groupoid is étale and Hausdorff. Let $\mathcal{A}$ be an atlas of a quasifold X. The structure groupoid **G** associated with the generating family of the atlas $\mathcal{A}$ is étale, namely: the projection $\mathrm{src}\colon \mathrm{Mor}(\mathbf{G}) \to \mathrm{Obj}(\mathbf{G})$ is an étale smooth map, that is, a local diffeomorphism at each point [**TB**, § 2.5].

Proposition 1. *For all* $\mathbf{g} \in \mathrm{Mor}(\mathbf{G})$*, there exists a D-open superset* $\mathcal{O}$ *of* $\mathbf{g}$ *such that* src *restricted to* $\mathcal{O}$ *is a local diffeomorphism.*

Proposition 2. *The groupoid* **G** *is locally compact and Hausdorff.*

[2]Since smooth maps are D-continuous and diffeomorphism are D-homeomorphisms.

Note. Since the atlas $\mathcal{A}$ is assumed to be locally finite, the preimages of the objects of $\mathbf{G}$ by the source map, or the target map, are countable.

218. MRW-equivalence of structure groupoids. We consider a quasifold X and two atlases $\mathcal{A}$ and $\mathcal{A}'$, with associated strict generating families $\mathcal{F}$ and $\mathcal{F}'$. We shall show in this section that the associated groupoids are equivalent in the sense of Muhly-Renault-Williams [**MRW87**, 2.1]; this will later give Morita-equivalent $\mathbf{C}^*$-algebras.

This section follows [**IZL18**, § 8]; we just check that the fact that the structure groups are countable and not just finite, does not change the result.

Let us recall what is an MRW-equivalence of groupoids. Let $\mathbf{G}$ and $\mathbf{G}'$ be two locally compact groupoids. We say that a locally compact space Z is a $(\mathbf{G}, \mathbf{G}')$-equivalence if

(i) Z is a left principal $\mathbf{G}$-space.
(ii) Z is a right principal $\mathbf{G}'$-space.
(iii) The $\mathbf{G}$ and $\mathbf{G}'$ actions commute.
(iv) The action of $\mathbf{G}$ on Z induces a bijection from $\mathrm{Z}/\mathbf{G}$ to $\mathrm{Obj}(\mathbf{G}')$.
(v) The action of $\mathbf{G}'$ on Z induces a bijection from $\mathrm{Z}/\mathbf{G}'$ to $\mathrm{Obj}(\mathbf{G})$.

Let $\mathrm{src}\colon \mathrm{Z} \to \mathrm{Obj}(\mathbf{G})$ and $\mathrm{trg}\colon \mathrm{Z} \to \mathrm{Obj}(\mathbf{G}')$ be the maps defining the composable pairs associated with the actions of $\mathbf{G}$ and $\mathbf{G}'$. That is, a pair (g, z) is composable if $\mathrm{trg}(g) = \mathrm{src}(z)$, and the composite is denoted by $g \cdot z$. Moreover, a pair (g', z) is composable if $\mathrm{src}(g') = \mathrm{trg}(z)$, and the composite is denoted by $z \cdot g'$.

Let us also recall that an action is principal in the sense of Muhly-Renault-Williams, if it is free: $g \cdot z = z$ only if g is a unit, and the *action map* $(g, z) \mapsto (g \cdot z, z)$, defined on the composable pairs, is proper [**MRW87**, § 2].

Now, using the hypothesis and notations of the previous pragraphs, let us define Z to be the space of germs of local diffeomorphisms, from the nebula of the family $\mathcal{F}$ to the nebula of the family $\mathcal{F}'$, that project on the identity by the evaluation map. That is,

$$Z = \left\{ \text{germ}(f)_r \;\middle|\; \begin{array}{l} f \in \text{Diff}_{\text{loc}}(\text{dom}(F), \text{dom}(F'), r \in \text{dom}(F), \\ F \in \mathcal{F}, F' \in \mathcal{F}' \text{ and } F' \circ f = F \restriction \text{dom}(f). \end{array} \right\}.$$

Let

$$\text{src}(\text{germ}(f)_r) = r \quad \text{and} \quad \text{trg}(\text{germ}(f)_r) = f(r).$$

For the sake of simplicity, we make an abuse of notation: in reality one should write, more precisely, $\text{src}(\text{germ}(f)_r) = (F, r)$ and $\text{trg}(\text{germ}(f)_r) = (F', f(r))$.

Then, the action of $g \in \text{Mor}(\mathbf{G})$ on $\text{germ}(f)_r$ is defined by composition if $\text{trg}(g) = r$, that is, $g \cdot \text{germ}(f)_r = \text{germ}(f \circ \varphi)_s$, where $g = \text{germ}(\varphi)_s$, $\varphi \in \text{Diff}_{\text{loc}}(\mathcal{N})$ and $\varphi(s) = r$. Symmetrically, the action of $g' \in \text{Mor}(\mathbf{G}')$ on $\text{germ}(f)_r$ is defined if $\text{src}(g') = f(r)$ by $z \cdot g' = \text{germ}(\varphi' \circ f)_r$, where $g' = \text{germ}(\varphi')_{f(r)}$. Then, we have:

Theorem. *The actions of $\mathbf{G}$ and $\mathbf{G}'$ on Z are principal, and Z is a $(\mathbf{G}, \mathbf{G}')$-equivalence in the sense of Muhly-Renault-Williams.*

Proof. First of all, let us point out that Z is a subspace of the morphisms of the groupoid $\mathbf{G}''$ built previously by adjunction of $\mathbf{G}$ and $\mathbf{G}'$, and is equipped with the subset diffeology. All these groupoids are locally compact and Hausdorff as we seen previously.

Let us check that the action of $\mathbf{G}$ on Z is free. In our case, $z = \text{germ}(f)_r$ and $g = \text{germ}(\varphi)_s$, where f and φ are local diffeomorphisms. If $g \cdot z = z$, then obviously $g = \text{germ}(1)_r$.

Next, let us denote by ρ the action of $\mathbf{G}$ on Z, defined on

$$\mathbf{G} \star Z = \{(g, z) \in \text{Mor}(\mathbf{G}) \times Z \mid \text{trg}(g) = \text{src}(z)\} \quad \text{by} \quad \rho(g, z) = g \cdot z.$$

This action is smooth because the composition of local diffeomorphisms is smooth, and passes onto the quotient groupoid in a smooth operation, see [IZL18, § 3]. Moreover, this action is invertible, its inverse being defined on

$$Z \star Z = \{(z', z) \in Z \times Z \mid \text{trg}(z') = \text{trg}(z)\}$$

by

$$\rho^{-1}(z', z) = (g = z' \cdot z^{-1}, z).$$

In detail, $\rho^{-1}(\mathrm{germ}(h)_s, \mathrm{germ}(f)_r) = (\mathrm{germ}(f^{-1} \circ h)_s, \mathrm{germ}(f)_r)$, with $f(r) = h(s)$. Now, the inverse is also smooth, when $Z \star Z \subset Z \times Z$ is equipped with the subset diffeology. In other words, ρ is an induction, that is, a diffeomorphism from $\mathbf{G}\star Z$ to $Z\star Z$. However, since $\mathbf{G}\star Z$ and $Z\star Z$ are defined by closed relations, and $\mathbf{G}$ and Z are Hausdorff, $\mathbf{G}\star Z$ and $Z\star Z$ are closed into their ambient spaces. Thus, the intersection of a compact subset in $Z\times Z$ with $Z\star Z$ is compact, and its preimage by the induction ρ is compact. Therefore, ρ is proper. We notice that the fact that the structure groups are no longer finite but just countable does not play a role here.

It remains to check that the action of $\mathbf{G}$ on Z induces a bijection of $Z/\mathbf{G}$ onto $\mathrm{Obj}(\mathbf{G}')$. Let us consider the map $\mathrm{class}\colon Z \to \mathrm{Obj}(\mathbf{G}')$ defined by $\mathrm{class}(\mathrm{germ}(f)_r) = f(r)$. Then, let $\mathrm{class}(z) = \mathrm{class}(z')$, with $z = \mathrm{germ}(f)_r$ and $z' = \mathrm{germ}(f')_{r'}$, that is, $f(r) = f'(r')$. However, since f and f' are local diffeomorphisms, $\varphi = f'^{-1} \circ f$ is a local diffeomorphism with $\varphi(r') = r$. Let $g = \mathrm{germ}(\varphi)_{r'}$, then $g \in \mathrm{Mor}(\mathbf{G})$ and $z' = g \cdot z$. Hence, the map class projects onto an injection from $Z/\mathbf{G}$ to $\mathrm{Obj}(\mathbf{G}')$. Now, let $(F', r') \in \mathrm{Obj}(\mathbf{G}')$, and let $x = F'(r') \in X$. Since $\mathcal{F}$ is a generating family, there exists $(F, r) \in \mathrm{Obj}(\mathbf{G})$ such that $F(r) = x$. Let ψ and ψ' be the charts of X defined by factorization: $F = \psi \circ \mathrm{class}$ and $F' = \psi' \circ \mathrm{class}'$, where $\mathrm{class}\colon \mathbf{R}^n \to \mathbf{R}^n/\Gamma$ and $\mathrm{class}'\colon \mathbf{R}^n \to \mathbf{R}^n/\Gamma'$. Let $\xi = \mathrm{class}(r)$ and $\xi' = \mathrm{class}'(r')$. Since $\psi(\xi) = \psi'(\xi') = x$, $\Psi =_{\mathrm{loc}} \psi'^{-1} \circ \psi$ is a local diffeomorphism from $\mathbf{R}^n/\Gamma$ to $\mathbf{R}^n/\Gamma'$ mapping ξ to ξ'. Hence, according to the lifting of local diffeomorphisms, there exists a local diffeomorphism f from $\mathrm{dom}(F)$ to $\mathrm{dom}(F')$, such that $\mathrm{class}' \circ f = \Psi \circ \mathrm{class}$ and $f(r) = r'$. Thus, $z = \mathrm{germ}(f)_r$ belongs to Z and $\mathrm{class}(z) = r'$ (precisely the element (F', r') of the nebula of $\mathcal{F}'$). Therefore, the injective map class from $Z/\mathbf{G}$ to $\mathrm{Obj}(\mathbf{G}')$ is also surjective, and identifies the two spaces. Obviously, what has been said for the side $\mathbf{G}$ can be translated to the side $\mathbf{G}'$; the construction is completely symmetric. In conclusion, Z satisfies the conditions of a $(\mathbf{G}, \mathbf{G}')$-equivalence, in the sense of Muhly-Renault-Williams. ▶

219. The C^*-algebra of a quasifold. Let X be a quasifold, let $\mathcal{A}$ be an atlas and let $\mathbf{G}$ be the structure groupoid associated with $\mathcal{A}$. Since the atlas $\mathcal{A}$ is locally finite, the convolution defined above is well defined. Indeed, in this case:

Proposition. *For every compactly supported complex function f on $\mathbf{G}$, for all $\nu = (F, r) \in \mathcal{N} = \mathrm{Obj}(\mathbf{G})$, the set of arrows $g \in \mathbf{G}^\nu$ such that $f(g) \neq 0$ is finite. That is, $\#\, \mathrm{Supp}(f \restriction \mathbf{G}^\nu) < \infty$. The convolution is then well defined on $\mathcal{C}(\mathbf{G})$.*

Then, for each atlas $\mathcal{A}$ of the quasifold X, we get the $\mathbf{C}^*$-algebra $\mathfrak{A} = (\mathcal{C}(\mathbf{G}), *)$. The dependence of the $\mathbf{C}^*$-algebra on the atlas is given by this theorem, which is a generalization of [**IZL18**, § 9] :

Theorem. *Different atlases give Morita-equivalent $\mathbf{C}^*$-algebras. Diffeomorphic quasifolds have Morita-equivalent $\mathbf{C}^*$-algebras.*

In other words, we have defined a functor from the subcategory of isomorphic {Quasifolds} in diffeology, to the category of Morita-equivalent {$\mathbf{C}^*$-algebras}.

220. Example: The C^*-algebra of the irrational torus. The first and famous example is the so-called Denjoy-Poincaré torus, or irrational torus, or noncommutative torus, or, more recently, quasitorus. It is, according to its first definition, the quotient set of the 2-torus T^2 by the irrational flow of slope $\alpha \in \mathbf{R} - \mathbf{Q}$. It is denoted by $T_\alpha = T^2/\mathcal{S}_\alpha$, where $\mathcal{S}_\alpha$ is the image of the line $y = \alpha x$ by the projection $\mathbf{R}^2 \to T^2 = \mathbf{R}^2/\mathbf{Z}^2$. Diffeologically speaking, $T_\alpha \simeq \mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z})$. The composite

$$\mathbf{R} \xrightarrow{\text{class}} \mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z}) \xrightarrow{f} T_\alpha, \quad \text{with } F = f \circ \text{class},$$

summarizes the situation where $\mathcal{A} = \{f : \mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z}) \to T_\alpha\}$ is the canonical atlas of T_α, containing the only chart f, and $\mathcal{F} = \{F = f \circ \text{class}\}$ is the associated canonical strict generating family. According to lifting the identity, the groupoid $\mathbf{G}_\alpha$ associated with the atlas $\mathcal{A}$ is simply

$$\mathrm{Obj}(\mathbf{G}_\alpha) = \mathbf{R} \quad \text{and} \quad \mathrm{Mor}(\mathbf{G}_\alpha) = \{(x, t_{n+\alpha m}) \mid x \in \mathbf{R} \text{ and } n, m \in \mathbf{Z}\}.$$

However, we can also identify T_α with $(\mathbf{R}/\mathbf{Z})/[(\mathbf{Z}+\alpha\mathbf{Z})/\mathbf{Z}]$, that is

$$T_\alpha \simeq S^1/\mathbf{Z}, \text{ with } \quad \underline{m}(z) = e^{2i\pi\alpha m}z,$$

for all $m \in \mathbf{Z}$ and $z \in S^1$. Moreover, the groupoid $\mathbf{S}$ of this action of $\mathbf{Z}$ on $S^1 \subset \mathbf{C}$ is simply

$$\mathrm{Obj}(\mathbf{S}_\alpha) = S^1 \quad \text{and} \quad \mathrm{Mor}(\mathbf{S}_\alpha) = \{(z, e^{2i\pi\alpha m}) \mid z \in S^1 \text{ and } m \in \mathbf{Z}\}.$$

The groupoids $\mathbf{G}_\alpha$ and $\mathbf{S}_\alpha$ are equivalent, thanks to the functor Φ from the first to the second:

$$\Phi_{\mathrm{Obj}}(x) = e^{2i\pi x} \quad \text{and} \quad \Phi_{\mathrm{Mor}}(x, t_{n+\alpha m}) = (e^{2i\pi x}, e^{2i\pi\alpha m}).$$

Moreover, they are also MRW-equivalent, by considering the set of germs of local diffeomorphisms $x \mapsto e^{2i\pi x}$, everywhere from $\mathbf{R}$ to S^1. Therefore, their associated $\mathbf{C}^*$-algebras are Morita equivalent. The algebra associated with $\mathbf{S}_\alpha$ has been computed numerous times and it is called *irrational rotation algebra* [MR81]. It is the universal $\mathbf{C}^*$-algebra generated by two unitary elements U and V, satisfying the relation

$$VU = e^{2i\pi\alpha}UV.$$

Remark 1. Thanks to the theorem on equivalence of diffeomorphic qusifolds, and because two irrational tori T_α and T_β are diffeomorphic if and only if α and β are conjugate modulo $GL(2,\mathbf{Z})$ [DI83], we get the corollary that, if α and β are conjugate modulo $GL(2,\mathbf{Z})$, then the algebra $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\beta$ are Morita equivalent, which is the direct sense of Rieffel's theorem [MR81, Thm 4].

Remark 2. The converse of Rieffel's theorem is a different matter altogether. We should recover a diffeological groupoid $\mathbf{G}_\alpha$ from the algebra $\mathfrak{A}_\alpha$. Then, the space of transitive components would be the required quasifold, as stated by the proposition on equivalence of structure groupoids. In the case of the irrational torus, it is not very difficult. The spectrum of the unitary operator V is the circle S^1 and the adjoint action by the operator U gives $UVU^{-1} = e^{2i\pi\alpha}V$, which translates on the spectrum by the irrational rotation of angle α. In that way, we recover the groupoid of the irrational rotations on the circle, which gives T_α as quasifold.

Remark 3. Of course, the situation of the irrational torus is simple and we do not exactly know how it can be reproduced for an arbitrary quasifold. However, this certainly is the way to follow to recover the quasifold from its algebra:

Task. Find the groupoid made with the Morita invariant of the algebra, which will give the space of transitivity components as the requested quasifold.

221. Example: The C*-algebra of R/Q. The diffeological space $\mathbf{R}/\mathbf{Q}$ is a legitimate quasifold. This is a simple example with a groupoid $\mathbf{G}$ given by

$$\mathrm{Obj}(\mathbf{G}) = \mathbf{R} \quad \text{and} \quad \mathrm{Mor}(\mathbf{G}) = \{(x, t_r) \mid x \in \mathbf{R} \text{ and } r \in \mathbf{Q}\}.$$

The algebra that is associated with $\mathbf{G}$ is the set $\mathfrak{A}$ of complex compact supported functions on $\mathrm{Mor}(\mathbf{G})$. Let us identify $\mathcal{C}^0(\mathrm{Mor}(\mathbf{G}), \mathbf{C})$ with $\mathrm{Maps}(\mathbf{Q}, \mathcal{C}^0(\mathbf{R}, \mathbf{C}))$ by

$$f = (f_r)_{r \in \mathbf{Q}} \quad \text{with} \quad f_r(x) = f(x, t_r), \quad \text{and let} \quad \mathrm{Supp}(f) = \{r \mid f_r \neq 0\}.$$

Then,

$$\mathfrak{A} = \left\{ f \in \mathrm{Maps}(\mathbf{Q}, \mathcal{C}^0(\mathbf{R}, \mathbf{C})) \mid \# \,\mathrm{Supp}(f) < \infty \right\}.$$

The convolution product and the algebra conjugation are, thus, given by:

$$(f * g)_r(x) = \sum_s f_{r-s}(x+s) g_s(x), \quad \text{and} \quad f_r^*(x) = f_{-r}(x+r)^*.$$

Now, the quotient $\mathbf{R}/\mathbf{Q}$ is also diffeomorphic to the $\mathbf{Q}$-circle

$$\mathrm{S}_\mathbf{Q} = \mathrm{S}^1/\mathcal{U}_\mathbf{Q}, \quad \text{where} \quad \mathcal{U}_\mathbf{Q} = \{e^{2i\pi r}\}_{r \in \mathbf{Q}}$$

is the subgroup of *rational roots of unity*. As a diffeological subgroup of S^1, $\mathcal{U}_\mathbf{Q}$ is discrete. The groupoid $\mathbf{S}$ of the action of $\mathcal{U}_\mathbf{Q}$ on S^1 is given by:

$$\mathrm{Obj}(\mathbf{S}) = \mathrm{S}^1 \quad \text{and} \quad \mathrm{Mor}(\mathbf{S}) = \left\{ (z, \tau) \;\middle|\; z \in \mathrm{S}^1 \text{ and } \tau \in \mathcal{U}_\mathbf{Q} \right\}.$$

The exponential $x \mapsto z = e^{2i\pi x}$ realizes a MRW-equivalence between the two groupoids $\mathbf{G}$ and $\mathbf{S}$. Their associated algebras are Morita-equivalent. The algebra $\mathfrak{S}$ associated with $\mathbf{S}$ is made of families of continuous complex functions indexed by rational roots of unity, in the same way as before:

$$\mathfrak{S} = \left\{ (f_\tau)_{\tau \in \mathcal{U}_\mathbf{Q}} \;\middle|\; f_\tau \in \mathcal{C}^0(\mathrm{S}^1, \mathbf{C}) \text{ and } \# \operatorname{Supp}(f) < \infty \right\}.$$

The convolution product and the algebra conjugation are, then, given by:

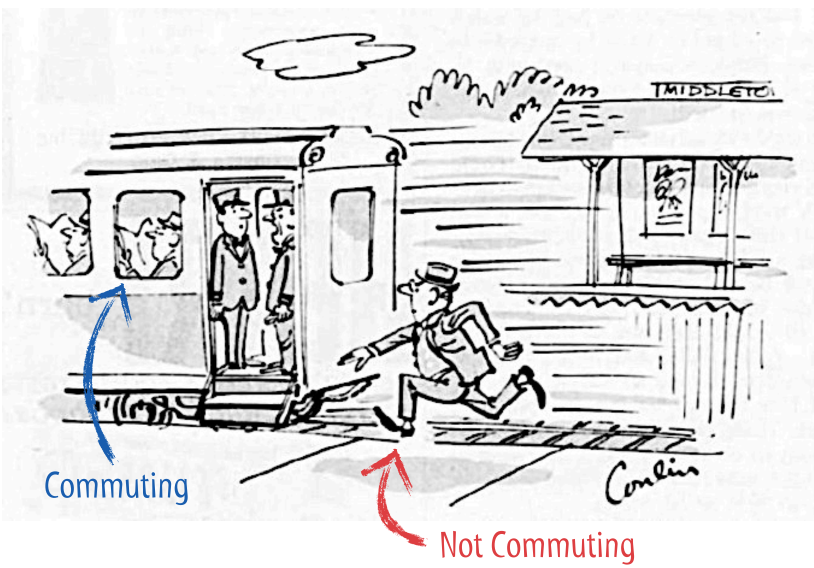
$$(f * g)_\tau(z) = \sum_\sigma f_{\bar\sigma\tau}(\sigma z) g_\sigma(z) \quad \text{and} \quad f^*_\tau(z) = f_{\bar\tau}(\tau z)^*,$$

where $\bar\tau = 1/\tau = \tau^*$, the complex conjugate.

Now, consider f and let $\mathcal{U}_p$ be the subgroup in $\mathcal{U}_\mathbf{Q}$ generated by $\operatorname{Supp}(f)$; this is the group of some root of unity ε of some order p. Let $\mathrm{M}_p(\mathbf{C})$ be the space of $p \times p$ matrices with complex coefficients. Define $f \mapsto \mathrm{M}$, from $\mathfrak{S}$ to $\mathrm{M}_p(\mathbf{C}) \otimes \mathcal{C}^0(\mathrm{S}^1, \mathbf{C})$, by

$$\mathrm{M}(z)^\sigma_\tau = f_{\bar\sigma\tau}(\sigma z), \quad \text{for all } z \in \mathrm{S}^1 \text{ and } \sigma, \tau \in \mathcal{U}_p.$$

That gives a representation of $\mathfrak{S}$ in the tensor product of the space of endomorphisms of the infinite-dimensional $\mathbf{C}$-vector space $\operatorname{Maps}(\mathcal{U}_\mathbf{Q}, \mathbf{C})$ by $\mathcal{C}^0(\mathrm{S}^1, \mathbf{C})$, with finite support.



Functional Diffeology on Fourier Coefficients

In this exercise we transfer the functional diffeology defined on smooth complex periodic functions to the space of Fourier coefficients of smooth complex periodic functions.

We denote by $\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$ the space of 1-periodic complex valued real functions,

$$\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C}) = \{f \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{C}) \mid f(x+1) = f(x)\}.$$

This space is then equipped with the functional diffeology. Let $f \in \mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$, and $(f_n)_{n\in \mathbf{Z}}$ be its sequence of Fourier coefficients

$$f_n = \int_0^1 f(x) e^{-2i\pi n x}\, dx, \;\; n \in \mathbf{Z}.$$

We know that the Fourier series converges to f, uniformly on $[0,1]$ (see for example [**Vil68**, § 2 Thm. 1]). We note

$$f(x) = \lim_{N\to\infty} \sum_{n=-N}^{+N} f_n\, e^{2i\pi n x} \quad \text{or} \quad f(x) = \sum_{n\in \mathbf{Z}} f_n\, e^{2i\pi n x}.$$

The set of Fourier coefficients of the smooth periodic real functions with values in $\mathbf{C}$ is a vector subspace $\mathcal{E}$ of $\mathrm{Maps}(\mathbf{Z}, \mathbf{C})$. Let

$$j : \mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C}) \to \mathrm{Maps}(\mathbf{Z}, \mathbf{C}) \quad \text{with} \quad j(f) = (f_n)_{n\in \mathbf{Z}}.$$

The map j is injective. We denote by $\mathcal{E}$ the subspace of $\mathrm{Maps}(\mathbf{Z}, \mathbf{C})$ made of the Fourier coefficients of the smooth periodic functions, that is,

$$\mathcal{E} = \Big\{ (f_n)_{n\in\mathbf{Z}} = j(f) \mid f \in \mathcal{C}^\infty_{\rm per}(\mathbf{R}, \mathbf{C}) \Big\}.$$

We know that the subspace $\mathcal{E}$ is made exactly of all rapidly decreasing sequences of complex numbers (op. cit.),

$$\text{for all } p \in \mathbf{N}, \quad n^p f_n \xrightarrow[|n|\to\infty]{} 0.$$

<u>222. **Exercise: Pushforward functional diffeology.**</u> Let $\mathcal{E}$ be the vector space of rapidly decreasing sequences of complex numbers, defined above.

Q1. Show that the parametrizations $\mathrm{P} : r \mapsto (f_n(r))_{n\in\mathbf{Z}}$ in $\mathcal{E}$ satisfying the following conditions define a diffeology.

(1) The functions $f_n : \mathrm{dom}(\mathrm{P}) \to \mathbf{C}$ are smooth.
(2) For all ball $\bar{\mathcal{B}} \subset \mathrm{dom}(\mathrm{P})$, for all $k, p \in \mathbf{N}$, there exists a positive number $\mathrm{M}_{k,p}$ such that for all integer n

$$\left| \frac{\partial^k f_n(r)}{\partial r^k} \right| \leq \frac{\mathrm{M}_{k,p}}{|n|^p} \quad \text{for all } r \in \mathcal{B}. \tag{$\clubsuit$}$$

Q2. Show that the diffeology defined by ($\clubsuit$) is the pushforward on $\mathcal{E}$, by j, of the functional diffeology on $\mathcal{C}^\infty_{\rm per}(\mathbf{R}, \mathbf{C})$.

<u>Note 1.</u> In other words, the parametrization $r \mapsto (f_n(r))_{n\in\mathbf{Z}}$ is a plot of the pushed forward functional diffeology if the functions f_n are smooth and their derivatives are uniformly rapidly decreasing, what we denote (rather imprecisely) by

$$n^p \frac{\partial^k f_n(r)}{\partial r^k} \xrightarrow[|n|\to\infty]{} 0.$$

<u>Note 2.</u> Thanks to paracompacity, it is enough that, for every point $r_0 \in \mathrm{dom}(\mathrm{P})$, there exists a ball $\mathcal{B}'$ centered at r_0 such that ($\clubsuit$) holds to ensure that ($\clubsuit$) holds on every ball $\bar{\mathcal{B}} \subset \mathrm{dom}(\mathrm{P})$.

☞ Proof. We verify, first of all, that the condition ($\clubsuit$) defines a diffeology on the space $\mathcal{E}$ of rapidly decreasing sequences of complex numbers.

A1 (Covering axiom). If $f_n(r)$ is constant in r, for all n, then the condition (♣) is trivially satisfied for $k > 0$, and for $k = 0$ it means that the series is rapidly decreasing, which is expected.

A2 (Locality axiom). According to Note 2, the condition (♣) is local.

A3 (Smooth compatibility axiom). Let $\mathrm{P} : (r \mapsto f_n(r))_{n\in\mathbf{Z}}$ satisfying (♣) and $\mathrm{F} : s \mapsto r$ be a smooth parametrization in the domain of P. We have, for all $k > 0$,

$$\frac{\partial^k f_n(s)}{\partial s^k} = \sum_{\ell=1}^{k} \frac{\partial^\ell f_n(r)}{\partial r^\ell} \cdot \mathrm{Q}_{k,\ell}\left(\frac{\partial r}{\partial s}, \ldots, \frac{\partial^k r}{\partial s^k}\right),$$

where the $\mathrm{Q}_{k,\ell}$ are polynomials. Now, since the function $s \mapsto r$ is smooth, the operators $\mathrm{Q}_{k,\ell}$ are bounded on every ball. Let then $s_0 \in \mathrm{dom}(\mathrm{F})$, $r_0 = \mathrm{F}(s_0)$ and $\mathcal{B}$ be a ball centered at r_0 such that the condition (♣) is satisfied. Let $\mathcal{B}' \subset \mathrm{F}^{-1}(\mathcal{B})$ be a ball centered at s_0, we have for all $s \in \mathcal{B}'$,

$$\begin{aligned}\left|\frac{\partial^k f_n(s)}{\partial s^k}\right| &\leq \sum_{\ell=1}^{k} \left|\frac{\partial^\ell f_n(r)}{\partial r^\ell}\right| \left|\mathrm{Q}_{k,\ell}\left(\frac{\partial r}{\partial s}, \ldots, \frac{\partial^k r}{\partial s^k}\right)\right| \\ &\leq \sum_{\ell=1}^{k} \frac{\mathrm{M}_{\ell,p}}{|n|^p} \sup_{s\in\mathcal{B}'} \left|\mathrm{Q}_{k,\ell}\left(\frac{\partial r}{\partial s}, \ldots, \frac{\partial^k r}{\partial s^k}\right)\right|,\end{aligned}$$

where the $\mathrm{M}_{\ell,p}$ are the constants of the inequality (♣) for the ball $\mathcal{B}$. Let then

$$m_{k,\ell} = \sup_{s\in\mathcal{B}'} \left|\mathrm{Q}_{k,\ell}\left(\frac{\partial r}{\partial s}, \ldots, \frac{\partial^k r}{\partial s^k}\right)\right| \text{ and } \mathrm{M}'_{k,p} = \sum_{\ell=1}^{k} m_{k,\ell}\, \mathrm{M}_{\ell,p},$$

we get, for all $s \in \mathcal{B}'$,

$$\left|\frac{\partial^k f_n(s)}{\partial s^k}\right| \leq \frac{\mathrm{M}'_{k,p}}{|n|^p}.$$

Hence, thanks to Note 2, $\mathrm{P} \circ \mathrm{F}$ still satisfies the condition (♣). Therefore, this condition defines a diffeology on the set $\mathcal{E}$ of rapidly decreasing sequences of complex numbers.

Let us show now that the diffeology defined on $\mathcal{E}$ by (♣) is the push-forward by j of the functional diffeology of $\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$.

1. Let us prove, first of all, that j is smooth, where $\mathcal{C}^\infty_{\rm per}(\mathbf{R},\mathbf{C})$ is equipped with the functional diffeology and $\mathcal{E}$ with the diffeology defined by (♣). Let $\mathrm{P} : r \to f_r$ be a plot of $\mathcal{C}^\infty_{\rm per}(\mathbf{R},\mathbf{C})$ defined on some real domain U, the composite $j \circ \mathrm{P}$ writes

$$j \circ \mathrm{P} : r \mapsto (f_n(r))_{n\in \mathbf{Z}} \quad \text{with} \quad f_n(r) = \int_0^1 f_r(x) e^{-2i\pi nx}\, dx.$$

Since $(r, x) \mapsto f_r(x)$ is smooth, the $f_n : r \mapsto f_n(r)$ are smooth, and:

$$\frac{\partial^k f_n(r)}{\partial r^k} = \int_0^1 \frac{\partial^k f_r(x)}{\partial r^k} e^{-2i\pi nx}\, dx.$$

For all nonzero integer n, after p integrations by parts, we get

$$\frac{\partial^k f_n(r)}{\partial r^k} = \frac{1}{(2i\pi n)^p} \int_0^1 \frac{\partial^p}{\partial x^p}\left(\frac{\partial^k f_r(x)}{\partial r^k}\right) e^{-2i\pi nx}\, dx.$$

Therefore, defining for every ball $\mathcal{B} \subset \mathrm{dom}(\mathrm{P})$ the number

$$\mathrm{M}_{k,p} = \frac{1}{(2\pi)^p} \sup_{r\in\mathcal{B}} \sup_{x\in[0,1]} \left| \frac{\partial^p}{\partial x^p} \frac{\partial^k}{\partial r^k} f_r(x)\right|,$$

we have

$$\left|\frac{\partial^k f_n(r)}{\partial r^k}\right| \leq \frac{\mathrm{M}_{k,p}}{|n|^p} \quad \text{for all } r \in \mathcal{B}.$$

Hence, $j \circ \mathrm{P}$ is a plot of $\mathcal{E}$. Therefore j is smooth.

2. Conversely, remember that j is injective, then let us show that $j^{-1} : \mathcal{E} \to \mathcal{C}^\infty_{\rm per}(\mathbf{R},\mathbf{C})$ is smooth. Let $\mathrm{P} : r \mapsto (f_n(r))_{n\in\mathbf{Z}}$ be a parametrization in $\mathcal{E}$ satisfying the condition (♣), and let

$$j^{-1} \circ \mathrm{P} : r \mapsto [x \mapsto f_r(x)].$$

The parametrization $r \mapsto f_r$ is given by

$$f_r(x) = \lim_{\mathrm{N}\to\infty} \mathrm{S_N}(r, x) \quad \text{with} \quad \mathrm{S_N}(r, x) = \sum_{n=-\mathrm{N}}^{+\mathrm{N}} f_n(r) e^{2i\pi nx}.$$

The $\mathrm{S_N}$ are smooth for all N, we want to show that the limit $(r, x) \mapsto f_r(x)$ is smooth too. For all $k, p, \mathrm{N} \in \mathbf{N}$, let us introduce the series of partial derivatives

$$S_N^{k,p}(r,x) = \frac{\partial^p}{\partial x^p}\frac{\partial^k}{\partial r^k}S_N(r,x) = \sum_{-N}^{+N}(2i\pi n)^p \frac{\partial^k f_n(r)}{\partial r^k} e^{2i\pi nx}.$$

The functions $S_N^{k,p}$ are smooth, with respect to (r,x), for all N, k, p. Let $|S|_N^{k,p}(r,x)$ be the series of moduli of the terms of $S_N^{k,p}$:

$$|S|_N^{k,p}(r,x) = \sum_{-N}^{+N} |2\pi n|^p \left|\frac{\partial^k f_n(r)}{\partial r^k}\right|.$$

Then, let $\bar{\mathcal{B}} \subset \mathrm{dom}(P)$ be some ball. Thanks to the hypothesis, applying (2) of **Q1** to $p+2$, we have, for all $r \in \mathcal{B}$,

$$\sum_{n=-N}^{+N} |2\pi n|^p \left|\frac{\partial^k f_n(r)}{\partial r^k}\right| \quad \leq \quad c + 2\sum_{n=1}^{N}(2\pi)^p \frac{M_{k,p+2}}{n^2},$$

where c, corresponding to $n = 0$, is some constant. Hence, for all $r \in \mathcal{B}$, $x \in [0,1]$ and $N \in \mathbf{N}$

$$|S|_N^{k,p}(r,x) \leq K$$

where $K = c + 2(2\pi)^p M_{k,p+2}(\pi^2/6)$. Next, since the series $|S|_N^{k,p}(r,x)$ is increasing and upper bounded, it is convergent, and therefore the series $S_N^{k,p}(r,x)$ is also convergent. Moreover, according to what preceded: the sequence of smooth functions $S_N^{k,p}$ converges uniformly on $\mathcal{B} \times [0,1]$ to some function $S_\infty^{k,p}$ when $N \to \infty$. Now, according to (an obvious improvement of) [**Don00**, Thm. 3.10] the map $(r,x) \mapsto f_r(x) = \lim_{N\to\infty} S_N(r,x)$ is smooth, and the $S_\infty^{k,p}$ are the partial derivatives

$$S_\infty^{k,p}(r,x) = \frac{\partial^p}{\partial x^p}\frac{\partial^k}{\partial r^k} f_r(x).$$

Thus, the parametrization $r \mapsto [x \mapsto f_r(x)]$ is a plot of $\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R},\mathbf{C})$ such that $j(r \mapsto f_r) = (f_n(r))_{n\in\mathbf{Z}}$. Therefore, j is a diffeomorphism from $\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R},\mathbf{C})$, equipped with the functional diffeology, to $\mathcal{E}$, equipped with the diffeology defined by (♣). In other words, the diffeology defined on $\mathcal{E}$ by (♣) is the pushforward of the functional diffeology on $\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R},\mathbf{C})$. ▶

Smooth Function on Periodic Functions

This exercise[1] gives a simple example of a function on the space of rapidly decreasing sequences that is smooth for the functional diffeology, inherited by the smooth periodic functions, but not smooth for the ordinary product diffeology.

We consider the subspace $\mathcal{E}$ of rapidly decreasing complex sequences $(z_n)_{n\in\mathbf{Z}} \in \mathcal{E}$. We consider the diffeology (♣) on $\mathcal{E}$, inherited by the functional diffeology on the space of smooth periodic functions, defined in "Functional diffeology on Fourier coefficients" previously. We denote by Can the diffeology inherited by the product diffeology, (♣) is finer than Can.

<u>**223. Exercise.**</u> Show that the linear map $\mathrm{F} : \mathcal{E} \to \mathbf{C}$,

$$\mathrm{F} : (z_n)_{n\in\mathbf{Z}} \mapsto \sum_{n\in\mathbf{Z}} z_n,$$

is smooth for the diffeology (♣), but not for the Can diffeology.

Hint: find a 1-plot $\gamma : t \mapsto (z_n(t))_{n\in\mathbf{Z}}$ for the diffeology Can such that $\mathrm{F} \circ \gamma$ is not smooth.

Proof. We know that the map $j : f \mapsto (f_n)_{n\in\mathbf{Z}}$, from $\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$ to $\mathcal{E}$, where the f_n are the Fourier coefficients of f, is a diffeomorphism when $\mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C})$ is equiped with the functional diffeology and $\mathcal{E}$ with the diffeology (♣) (op. cit.). The inverse is given by $j^{-1} : (f_n)_{n\in\mathbf{Z}} \mapsto$

[1]Joint with P. Donato.

$[x \mapsto \sum_{n\in\mathbf{Z}} f_n e^{2i\pi nx}]$. Let $\zeta = j^{-1}(z_n)_{n\in\mathbf{Z}}$, then $\mathrm{F}((z_n)_{n\in\mathbf{Z}}) = \zeta(0)$. Therefore $\mathrm{F} = \hat{0} \circ j^{-1}$ where $\hat{0}$ is the evaluation at the origin, $\hat{0}(f) = f(0)$. Since $\hat{0} : \mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C}) \to \mathbf{C}$ is smooth for the functional diffeology and j^{-1} is a diffeomorphism, F is smooth.

Next, consider the path

$$\gamma : t \mapsto (z_n(t))_{n\in\mathbf{Z}} \quad \text{with} \quad z_n(t) = e^{-|n|} e^{ie^{2|n|}t},$$

where $t \in \mathbf{R}$. Since every z_n is smooth, the path γ is smooth with $\mathcal{E} \subset \prod_{n\in\mathbf{Z}} \mathbf{C}$ equipped with the subset diffeology of the product diffeology. Since $|z_n(t)| = e^{-|n|}$ is rapidly decreasing in n, the partial sums $\sum_{n=-\mathrm{N}}^{\mathrm{N}} z_n(t)$ converge for all $t \in \mathbf{R}$. Let $f = \mathrm{F} \circ \gamma$, that is,

$$f(t) = \sum_{n\in\mathbf{Z}} e^{-|n|} e^{ie^{2|n|}t}.$$

We shall check now that f is not derivable at $t = 0$. Consider the variation

$$\begin{aligned} \Delta f(t, 0) &= \frac{f(t) - f(0)}{t} \\ &= \sum_{n\in\mathbf{Z}} \frac{e^{-|n|} e^{ie^{2|n|}t} - e^{-|n|}}{t} \\ &= \sum_{n\in\mathbf{Z}} e^{-|n|} \times \frac{e^{ie^{2|n|}t} - 1}{t}. \end{aligned}$$

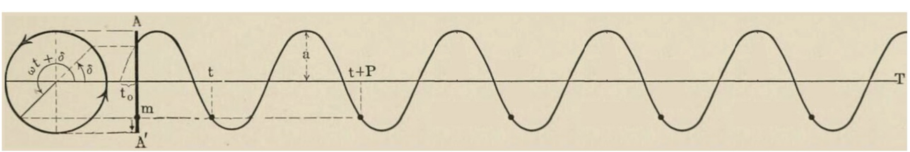
But,

$$\frac{e^{ie^{2|n|}t} - 1}{t} \xrightarrow[t\to 0]{} ie^{2|n|}.$$

Hence,

$$\Delta f(t, 0) \xrightarrow[t\to 0]{} i \sum_{n\in\mathbf{Z}} e^{|n|},$$

but that is not convergent. Therefore f is not derivable in $t = 0$. ▶



Symplectic Diffeology on the Space of Smooth Periodic Functions

We consider a symplectic structure on the space of smooth periodic functions and the action of the infinite torus as we have seen in previous notes "Functional Diffeology on Fourier Coefficients" and "Infinite Torus Action on Smooth Periodic Functions". We study the action of the irrational solenoid and we show a reduction process for its moment map levels, extending a diffeological version of Elisa Prato's quasispheres construction [**EP01**] to infinite dimension,[1] using the diffeology tools.

54. A primitive of the symplectic form

We consider the space $\mathcal{C}^\infty_{\rm per}(\mathbf{R},\mathbf{C})$ of 1-periodic complex valued real functions equipped with the functional diffeology. We denote by $\mathcal{E}$ the space of Fourier coefficients of smooth periodic functions and by $j : \mathcal{C}^\infty_{\rm per}(\mathbf{R},\mathbf{C}) \to \mathcal{E}$ the mapping

$$j(f) = (f_n)_{n\in\mathbf{Z}} \quad \text{with} \quad f_n = \int_0^1 f(x)e^{-2i\pi nx}dx,$$

[1]This exercise has been inspired by a few and very interesting exchanges I have had with Elisa Prato and Fiametta Battaglia in January 2014 at Firenze where I stayed at the invitation of Elisa. Let me take this opportunity to thank them sincerely.

and $\mathcal{E}$ is equipped with the pushforward by j of the functional diffeology on $\mathcal{C}^\infty_{\rm per}(\mathbf{R}, \mathbf{C})$ from "Functional Diffeology on Fourier Coefficients".

<u>**224. Exercise I.**</u> For all plots $\mathrm{P} : \mathrm{U} \to \mathcal{C}^\infty_{\rm per}(\mathbf{R}, \mathbf{C})$ let

$$\varepsilon(\mathrm{P})_r(\delta r) = \frac{1}{2i\pi} \int_0^1 \overline{f_r(x)}\, \frac{\partial f_r(x)}{\partial r}(\delta r)\, dx.$$

Q1. Check that ε defines a differential 1-form on $\mathcal{C}^\infty_{\rm per}(\mathbf{R}, \mathbf{C})$.

Q2. Develop ε and $\omega = d\varepsilon$ also in terms of the real and imaginary parts $f(x) = a(x) + ib(x)$.

Q3. Develop ε and ω in terms of Fourier coefficients.

Proof. Let us check first that ε is a well defined form. Let $\mathrm{P} : r \mapsto f_r$ be a plot in $\mathcal{C}^\infty_{\rm per}(\mathbf{R}, \mathbf{C})$, let $\mathrm{Q} : s \mapsto r$ be a plot in $\mathrm{dom}(\mathrm{P})$. We want to check that $\varepsilon(\mathrm{P} \circ \mathrm{Q}) = \mathrm{Q}^*(\varepsilon(\mathrm{P}))$, that is,

$$\varepsilon(\mathrm{P} \circ \mathrm{Q})_s(\delta s) = \varepsilon(\mathrm{P})_r(\delta r) \quad \text{with} \quad r = \mathrm{Q}(s) \quad \text{and} \quad \delta r = \frac{\partial r}{\partial s}(\delta s),$$

where δr is a tangent vector at r to $\mathrm{dom}(\mathrm{P})$. But

$$\begin{aligned}
\varepsilon(\mathrm{P} \circ \mathrm{Q})_s(\delta s) &= \frac{1}{2i\pi} \int_0^1 \overline{\mathrm{P} \circ \mathrm{Q}(s)(x)}\, \frac{\partial \mathrm{P} \circ \mathrm{Q}(s)(x)}{\partial s}(\delta s)\, dx \\
&= \frac{1}{2i\pi} \int_0^1 \overline{\mathrm{P}(r)(x)}\, \frac{\partial \mathrm{P}(r)(x)}{\partial r} \frac{\partial r}{\partial s}(\delta s)\, dx, \text{ with } r = \mathrm{Q}(s) \\
&= \frac{1}{2i\pi} \int_0^1 \overline{f_r(x)}\, \frac{\partial f_r(x)}{\partial r}(\delta r)\, dx, \text{ with } \delta r = \frac{\partial r}{\partial s}(\delta s) \\
&= \varepsilon(\mathrm{P})_r(\delta r), \text{ with } r = \mathrm{Q}(s) \text{ and } \delta r = \frac{\partial r}{\partial s}(\delta s).
\end{aligned}$$

Therefore, $\varepsilon(\mathrm{P} \circ \mathrm{Q}) = \mathrm{Q}^*(\varepsilon(\mathrm{P}))$ and ε is a well defined differential 1-form on $\mathcal{C}^\infty_{\rm per}(\mathbf{R}, \mathbf{C})$.

For Question 2. The exterior derivative of ε is given by an immediate application of the definition

$$d\varepsilon(\mathrm{P})_r(\delta_r, \delta' r) = \delta(\varepsilon(\mathrm{P})_r(\delta' r)) - \delta'(\varepsilon(\mathrm{P})_r(\delta r)),$$

for two independent variations δ and δ'. Then,

$$\begin{aligned}\omega(\mathrm{P})_r(\delta_r, \delta' r) &= \frac{1}{2i\pi}\int_0^1 \overline{\frac{\partial f_r(x)}{\partial r}}(\delta r)\frac{\partial f_r(x)}{\partial r}(\delta' r) \\ &- \overline{\frac{\partial f_r(x)}{\partial r}}(\delta' r)\frac{\partial f_r(x)}{\partial r}(\delta r)\, dx.\end{aligned}$$

Now let $f_r(x) = a_r(x) + ib_r(x)$ and let us denote simply

$$a = a_r(x),\ \ b = b_r(x),\ \ \delta a = \frac{\partial a}{\partial r}(\delta r),\ \text{ and } \delta b = \frac{\partial b}{\partial r}(\delta r).$$

Then,

$$\begin{aligned}\varepsilon(\mathrm{P})_r(\delta r) &= \frac{1}{2i\pi}\int_0^1 \overline{f_r(x)}\,\frac{\partial f_r(x)}{\partial r}(\delta r)\, dx \\ &= \frac{1}{2i\pi}\int_0^1 (a - ib)(\delta a + i\delta b)\, dx \\ &= \frac{1}{2\pi}\int_0^1 (a\delta b - b\delta a)\, dx + \frac{1}{2i\pi}\int_0^1 (a\delta a + b\delta b)\, dx \\ &= \frac{1}{2\pi}\int_0^1 (a\delta b - b\delta a)\, dx + \frac{1}{4i\pi} d\left[\int_0^1 (a^2 + b^2)\, dx\right](\delta r)\end{aligned}$$

Therefore, since $d \circ d = 0$, the exterior derivative is

$$\omega(\mathrm{P})_r(\delta r, \delta' r) = \frac{1}{\pi}\int_0^1 (\delta a\, \delta' b - \delta' a\, \delta b)\, dx.$$

For Question 3. Since everything is smooth we can exchange limits and integrals. Let

$$f_n(r) = \int_0^1 f_r(x) e^{-2i\pi n x} dx,$$

we have then:

$$\begin{aligned}\varepsilon(\mathrm{P})_r(\delta r) &= \frac{1}{2i\pi}\int_0^1 \overline{f_r(x)}\,\frac{\partial f_r(x)}{\partial r}(\delta r)\, dx. \\ &= \frac{1}{2i\pi}\int_0^1 \left(\sum_{n\in\mathbf{Z}} \overline{f_n(r)} e^{-2i\pi n x}\right)\left(\sum_{m\in\mathbf{Z}} \frac{\partial f_m(r)}{\partial r}(\delta r) e^{2i\pi m x}\right) dx \\ &= \frac{1}{2i\pi}\sum_{n,m\in\mathbf{Z}} \overline{f_n(r)}\,\frac{\partial f_m(r)}{\partial r}(\delta r)\int_0^1 e^{2i\pi(m-n)x}\, dx\end{aligned}$$

$$= \frac{1}{2i\pi} \sum_{n \in \mathbf{Z}} \overline{f_n(r)} \frac{\partial f_n(r)}{\partial r}(\delta r).$$

And therefore, for the exterior derivative,

$$\begin{aligned} \omega(\mathrm{P})_r(\delta r, \delta' r) &= \frac{1}{2i\pi} \sum_{n \in \mathbf{Z}} \overline{\frac{\partial f_n(r)}{\partial r}}(\delta r) \frac{\partial f_n(r)}{\partial r}(\delta' r) \\ &- \overline{\frac{\partial f_n(r)}{\partial r}}(\delta' r) \frac{\partial f_n(r)}{\partial r}(\delta r). \end{aligned}$$

Let us notice that if $\hat{x} : \mathcal{C}^\infty_{\mathrm{per}}(\mathbf{R}, \mathbf{C}) \to \mathbf{C}$ denotes the *evaluation map* $\hat{x}(f) = f(x)$, then the 2-form ω is the mean value of the pullbacks

$$\omega = \frac{1}{\pi} \int_0^1 \hat{x}^*(\omega_0)\, dx,$$

where ω_0 is the canonical symplectic form on $\mathbf{C}$. ▶

55. Hamiltonian action of the infinite torus

225. Exercise II. We consider the 2-form $\omega = d\varepsilon$ defined on the space of Fourier coefficients $\mathcal{E}$, equipped with the pushforward of the functional diffeology. We consider then the action of the infinite torus T^∞ on $\mathcal{E}$, described in "Infinite Torus Action on Smooth Periodic Functions".

$$(\tau_n)_{n \in \mathbf{Z}} \cdot (\mathrm{Z}_n)_{n \in \mathbf{Z}} = (\tau_n \mathrm{Z}_n)_{n \in \mathbf{Z}}.$$

Q1. Verify that the action of T^∞ on $\mathcal{E}$ is Hamiltonian and exact.[2]

Q2. Show that the moment maps of the action of T^∞ on $\mathcal{E}$ are given by

$$\mu(\mathrm{Z}) = \frac{1}{2i\pi} \sum_{n \in \mathbf{Z}} |\mathrm{Z}_n|^2\, \pi_n^*(\theta) + \sigma,$$

where $\mathrm{Z} = (\mathrm{Z}_n)_{n \in \mathbf{Z}} \in \mathcal{E}$, $\pi_n : \mathrm{T}^\infty \to \mathrm{U}(1)$ is the n-th projection $\pi_n(\mathrm{Z}) = \mathrm{Z}_n$, θ is the canonical invariant 1-form on $\mathrm{U}(1)$, and σ is a constant momentum of T^∞, that is, a constant invariant 1-form.

[2] See [TB, Chapter 9]

Q3. Let $(\alpha_n)_{n\in\mathbf{Z}}$ be a sequence of irrational numbers independent over $\mathbf{Q}$, see Exercise II of "Infinite Torus Action on Smooth Periodic Functions". Consider the induction

$$\iota : \mathbf{R} \to \mathrm{T}^\infty \quad \text{with} \quad \iota(t) = \left(e^{2i\pi\alpha_n t}\right)_{n\in\mathbf{Z}},$$

and the induced action of $\mathbf{R}$ on $\mathcal{E}$

$$\underline{t}(\mathrm{Z}_n)_{n\in\mathbf{Z}} = \left(e^{2i\pi\alpha_n t}\mathrm{Z}_n\right)_{n\in\mathbf{Z}}.$$

Show that the 1-point moment maps are given by

$$\nu(\mathrm{Z}) = h(\mathrm{Z})\,dt \quad \text{with} \quad h(\mathrm{Z}) = \sum_{n\in\mathbf{Z}} \alpha_n |\mathrm{Z}_n|^2 + c,$$

where c is some constant.

Proof. For Question 1. The primitive ε is invariant by the action of T^∞. Let us recall that

$$\varepsilon(\mathrm{P})_r(\delta r) = \frac{1}{2i\pi}\sum_{n\in\mathbf{Z}} \overline{\mathrm{Z}_n(r)}\,\frac{\partial \mathrm{Z}_n(r)}{\partial r}(\delta r),$$

for all plots $\mathrm{P} : r \mapsto (\mathrm{Z}_n(r))_{n\in\mathbf{Z}}$ in $\mathcal{E}$. Then, let $\tau = (\tau_n)_{n\in\mathbf{Z}} \in \mathrm{T}^\infty$, we have

$$\begin{aligned}
\underline{\tau}^*(\varepsilon)(\mathrm{P})_r(\delta r) &= \varepsilon(\underline{\tau}\circ \mathrm{P})_r(\delta r)\\
&= \frac{1}{2i\pi}\sum_{n\in\mathbf{Z}} \overline{\tau_n\,\mathrm{Z}_n(r)}\,\frac{\partial \tau_n\,\mathrm{Z}_n(r)}{\partial r}(\delta r)\\
&= \frac{1}{2i\pi}\sum_{n\in\mathbf{Z}} \overline{\mathrm{Z}_n(r)}\,\bar\tau_n\tau_n\,\frac{\partial \mathrm{Z}_n(r)}{\partial r}(\delta r)\\
&= \frac{1}{2i\pi}\sum_{n\in\mathbf{Z}} \overline{\mathrm{Z}_n(r)}\,\frac{\partial \mathrm{Z}_n(r)}{\partial r}(\delta r)\\
&= \varepsilon(\mathrm{P})_r(\delta r).
\end{aligned}$$

Therefore, the action is Hamiltonian and equivariant [TB, § 9.11].

For Question 2. Let θ be the canonical 1-form on $\mathrm{U}(1)$ defined by

$$\mathrm{class}^*(\theta) = dt, \quad \text{with} \quad \mathrm{class} : \mathbf{R} \to \mathrm{T} = \mathbf{R}/\mathbf{Z}.$$

Let $\pi_n : \mathrm{T}^\infty \to \mathrm{U}(1)$ be the n-th projection $\mathrm{Z} \mapsto \mathrm{Z}_n$, for all $\mathrm{Z} = (\mathrm{Z}_n)_{n\in\mathbf{Z}} \in \mathcal{E}$. Let $\zeta : r \mapsto (\zeta_n(r))_{n\in\mathbf{Z}}$ be a plot in T^∞, the 1-point moment map μ of the action of T^∞ on $\mathcal{E}$ is given, up to a constant, by

$$\begin{aligned}
\mu(Z)(\zeta)_r(\delta r) &= \hat{Z}^*(\varepsilon)(\zeta)_r(\delta r) \\
&= \varepsilon(\hat{Z} \circ \zeta)_r(\delta r) \\
&= \frac{1}{2i\pi} \sum_{n \in Z} \overline{Z_n \zeta_n(r)} \, \frac{\partial Z_n \zeta_n(r)}{\partial r}(\delta r) \\
&= \frac{1}{2i\pi} \sum_{n \in Z} |Z_n|^2 \, \bar{\zeta}_n(r) \, \frac{\partial \zeta_n(r)}{\partial r}(\delta r) \\
&= \frac{1}{2i\pi} \sum_{n \in Z} |Z_n|^2 \, \pi_n^*(\theta)(\zeta)_r(\delta r).
\end{aligned}$$

Therefore, a general 1-point moment map writes

$$\mu(Z) = \frac{1}{2i\pi} \sum_{n \in Z} |Z_n|^2 \, \pi_n^*(\theta) + \sigma.$$

Remark that on any ball, for all $n \in \mathbf{Z}$, $\zeta_n(r) = \exp(2i\pi \, t_n(r))$ for some smooth real functions t_n. Then, the moment map is also given, modulo a constant, by

$$\mu(Z)(\zeta)_r(\delta r) = \sum_{n \in Z} |Z_n|^2 \, \frac{\partial t_n(r)}{\partial r}(\delta r).$$

Note also that, since ζ is a plot in T^∞ for the tempered diffeology, the norm of the derivatives $\partial \zeta_n(r)/\partial r$, that is, $\partial t_n(r)/\partial r$, are dominated by a polynomial in n what insures the convergence of the series defining the moment map just above.

For Question 3. The induction $\iota : \mathbf{R} \to T^\infty$

$$\iota : t \mapsto \left(e^{2i\pi\alpha_n t}\right)_{n \in Z}$$

induces a projection $\iota^* : \mathcal{T}^*_\infty \to \mathbf{R}^*$, where $\mathcal{T}^*_\infty$ is the space of momenta of T^∞. The moment maps with respect to the group $\mathbf{R}$ are then the composites $\nu = \iota^* \circ \mu$, that is,

$$\begin{aligned}
\nu &= \iota^* \left\{ \frac{1}{2i\pi} \sum_{n \in Z} |Z_n|^2 \, \pi_n^*(\theta) + \sigma \right\} \\
&= \frac{1}{2i\pi} \sum_{n \in Z} |Z_n|^2 \, (\pi_n \circ \iota)^*(\theta) + \iota^*(\sigma).
\end{aligned}$$

But

$$\pi_n \circ \iota : t \mapsto \exp(2i\pi\alpha_n t), \quad \text{then} \quad (\pi_n \circ \iota)^*(\theta) = 2i\pi\alpha_n \, dt.$$

Thus,

$$\nu(Z) = h(Z)\, dt \quad \text{with} \quad h(Z) = \sum_{n \in \mathbf{Z}} \alpha_n |Z_n|^2 + c,$$

where $\iota^*(\sigma) = c\, dt$, $c \in \mathbf{R}$. ▶

56. Orbits of the Hamiltonian flow

<u>226. Exercise III.</u> We continue with the data of previous exercise. Let Y be a level of the moment map ν of the action of $\mathbf{R}$, that is,

$$Y = \Big\{ Z = (Z_n)_{n \in \mathbf{Z}} \in \mathcal{E} \mid \sum_{n \in \mathbf{Z}} \alpha_n |Z_n|^2 = c \Big\} \quad \text{with } c > 0.$$

Q1. Verify that, for all $Z \in Y$, if there exists $Z_n \neq 0$ and $Z_m \neq 0$, then the stabilizer of Z is reduced to $\{0\}$ and the orbit of Z by $\mathbf{R}$, equipped with the subset diffeology, is diffeomorphic to $\mathbf{R}$. Such orbits are called *principal orbits*.

Q2. Verify that the non principal orbits, that is, the *singular orbits*, are the subspaces

$$S^1_n = \{Z \in Y \mid Z_m = 0 \text{ if } m \neq n\}, \quad \text{with} \quad n \in \mathbf{N},$$

each diffeomorphic to the circle S^1.

Q3. Show that the union

$$\mathcal{S} = \bigcup_{n \in \mathbf{Z}} S^1_n \subset Y,$$

equipped with the subset diffeology is actually the sum of the S^1_n [**TB**, § 1.39], that is,

$$\mathcal{S} = \coprod_{n \in \mathbf{Z}} S^1_n \quad \text{and then} \quad \dim(\mathcal{S}) = 1.$$

Q4. Show that $Y - \mathcal{S}$ is open for the D-topology (D-open) [**TB**, § 2.8].

Proof. For Question 1. Let $Z \in Y$ with $Z_n \neq 0$ and $Z_m \neq 0$, the map

$$t \mapsto (e^{2i\pi\alpha_n t}\tau_n, e^{2i\pi\alpha_m t}\tau_m) \quad \text{with} \quad \begin{cases} \tau_n = \dfrac{Z_n}{|Z_n|} \\ \tau_m = \dfrac{Z_m}{|Z_m|} \end{cases}$$

is an induction from $\mathbf{R}$ into T^2, because α_n and α_m are independent over $\mathbf{Q}$, see [**TB**, Exercise 31]. Therefore, the orbit map $t \mapsto \underline{t}(Z)$ is an induction.

For Question 2. There exists n such that for all $m \neq n$, $Z_m = 0$ but $Z_n \neq 0$, since $\sum_{n\in \mathbf{Z}} \alpha_n |Z_n|^2 > 0$. The orbit map is a covering onto the circle S^1_n induced in Y.

For Question 3. Let $P : U \to \mathcal{S}$ be a plot. For every $n \in \mathbf{Z}$ let $\mathcal{O}_n = (\pi_n \circ P)^{-1}(\mathbf{C} - \{0\})$, where $\pi_n : Y \to \mathbf{C}$ is the projection $\pi_n((Z_m)_{m\in \mathbf{Z}}) = Z_n$. Since $\pi_n \circ P$ is smooth, then continuous, every $\mathcal{O}_n \subset U$ is open. Moreover, let $n \neq m$, assume $r \in \mathcal{O}_n \cap \mathcal{O}_m$, that is, $Z_n(r) \neq 0$ and $Z_m(r) \neq 0$, but P takes its values in the union of the S^1_m, $m \in \mathbf{Z}$, hence $Z_n(r) \neq 0$ implies $Z_m(r) = 0$ for all $m \neq n$ thus $\mathcal{O}_n \cap \mathcal{O}_m = \varnothing$. Therefore,

$$U = \bigcup_{n\in \mathbf{Z}} \mathcal{O}_n \quad \text{and} \quad \mathcal{O}_n \cap \mathcal{O}_m = \varnothing,$$

for all $n \neq m$.

That means that the $\mathcal{O}_n$ are the connected components of U. Thus, P takes locally its values in the S^1_n, $n \in \mathbf{Z}$, that means that $\mathcal{S}$ is the diffeological sum of the circles S^1_n, $n \in \mathbf{Z}$, see [**TB**, § 1.39].

For Question 4. Let $P : U \to Y$ be a plot, $P(r) = (Z_n(r))_{n\in \mathbf{Z}}$. For all $r_0 \in P^{-1}(Y - \mathcal{S})$ there exist at least two different indices n and m such that $Z_n(r_0) \neq 0$ and $Z_m(r_0) \neq 0$. Since Z_n and Z_m are smooth there exists an open neighborhood V of r_0 such that $Z_n(r) \neq 0$ and $Z_m(r) \neq 0$, for all $r \in V$, that is, $V \subset P^{-1}(Y - \mathcal{S})$. Thus, $P^{-1}(Y - \mathcal{S})$ is a union of open domains, it is then an open domain, and consequently $Y - \mathcal{S}$ is D-open. ▶

57. Reduction of a moment map level

<u>**227. Exercise IV.**</u> We denote by X the space of orbits of the action of $\mathbf{R}$ on Y. We equip Y with the quotient diffeology, and denote by $\mathrm{pr} : \mathrm{Y} \to \mathrm{X}$ the projection.

Q1. Why could we call the space X, an *infinite quasiprojective space*?

Q2. Show that there exists a closed 2-form ϖ on X (We say that ϖ is parasymplectic) such that

$$\omega \restriction \mathrm{Y} = \mathrm{pr}^*(\varpi).$$

This is a particular case of symplectic reduction with singularities.

Proof. For Question 1. Let $\mathrm{Z} = (\mathrm{Z}_n)_{n \in \mathbf{Z}} \in \mathrm{Y}$, by the change $\mathrm{Z}_n \mapsto \sqrt{\alpha_n}\mathrm{Z}_n/\sqrt{c}$ the subspace Y is mapped into $\mathrm{S}^\infty \in \mathcal{E}$, the unit sphere in $\mathcal{E}$. Now, if all α_n would be equal to 1 then the action of $\mathbf{R}$ would be the action of S^1 and X would be diffeomorphic to $\mathbf{CP}^\infty$, the infinite projective space, and the projection $\mathrm{pr} : \mathrm{Y} \to \mathrm{X}$ would be the infinite Hopf fibration, see [TB, § 4.11] for the same infinite projective space with another diffeology. That explains the choice of vocabulary.

For Question 2. We shall apply the general criterion for a differential form to be the pullback of another one. Let $\mathrm{P} : \mathrm{U} \to \mathrm{Y}$ and $\mathrm{P}' : \mathrm{U} \to \mathrm{Y}$ be two plots

$$\begin{array}{ccc} \mathrm{U} & \overset{\mathrm{P}}{\underset{\mathrm{P}'}{\rightrightarrows}} & \mathrm{Y} \\ & & \Big\downarrow \mathrm{pr} \\ & & \mathrm{X} \end{array} \qquad \text{such that} \qquad \mathrm{pr} \circ \mathrm{P} = \mathrm{pr} \circ \mathrm{P}'.$$

We want to check if, in these conditions, $\omega(\mathrm{P}) = \omega(\mathrm{P}')$. That would insure the existence of ϖ, a (necessarily closed) 2-form on X such that $\omega = \mathrm{pr}^*(\varpi)$ [TB, § 6.38]. We consider first of all what happens on the open subset

$$\mathrm{U}_0 = \mathrm{P}^{-1}(\mathrm{Y} - \mathcal{S}).$$

Since $\mathrm{pr} \circ \mathrm{P} = \mathrm{pr} \circ \mathrm{P}'$, $\mathrm{P}^{-1}(\mathrm{Y} - \mathcal{S}) = \mathrm{P}'^{-1}(\mathrm{Y} - \mathcal{S}) = \mathrm{U}_0$. Now, the restrictions of P and P′ on U_0 take their values in the subset of Y made of principal orbits of **R**, for which the stabilizer of the action of **R** is $\{0\}$. Therefore, for each $r \in \mathrm{U}_0$ there is a unique $\tau(r) \in \mathbf{R}$ such that, for all n, $\mathrm{Z}'_n(r) = e^{2i\pi\alpha_n\tau(r)}\mathrm{Z}_n(r)$. Now, $\omega = d\varepsilon$, and

$$\begin{aligned}
\varepsilon(\mathrm{P}')_r(\delta r) &= \frac{1}{2i\pi}\sum_{n\in\mathbf{Z}} \bar{\mathrm{Z}}'_n(r)\frac{\partial \mathrm{Z}'_n(r)}{\partial r}(\delta r) \\
&= \frac{1}{2i\pi}\sum_{n\in\mathbf{Z}} \bar{\mathrm{Z}}_n(r)\frac{\partial \mathrm{Z}_n(r)}{\partial r}(\delta r) \\
&+ \Big(\sum_{n\in\mathbf{Z}} \alpha_n \bar{\mathrm{Z}}_n(r)\mathrm{Z}_n(r)\Big)\frac{\partial \tau(r)}{\partial r}(\delta r) \\
&= \varepsilon(\mathrm{P})_r(\delta r) + c\,\tau^*(dt).
\end{aligned}$$

Therefore, $[\omega(\mathrm{P}')-\omega(\mathrm{P})] \upharpoonright \mathrm{U}_0 = 0$. Thus, by continuity, $[\omega(\mathrm{P}')-\omega(\mathrm{P})] \upharpoonright \bar{\mathrm{U}}_0 = 0$, where $\bar{\mathrm{U}}_0$ is the closure of U_0. It remains to check what happens on the complementary $\mathrm{V} = \mathrm{U} - \bar{\mathrm{U}}_0$. The subset V is open, thus $\mathrm{P} \upharpoonright \mathrm{V}$ and $\mathrm{P}' \upharpoonright \mathrm{V}$ are two plots of Y but with values in the subset of singular orbits $\mathcal{S}$. Since $\mathcal{S}$ has dimension 1 and ω is a 2-form, $\omega(\mathrm{P} \upharpoonright \mathrm{V}) = \omega(\mathrm{P}' \upharpoonright \mathrm{V}) = 0$. In conclusion $\omega(\mathrm{P}') = \omega(\mathrm{P})$ everywhere on U. That proves that there exists a 2-form ϖ on $\mathrm{X} = \mathrm{Y}/\mathbf{R}$ such that $\mathrm{pr}^*(\varpi) = \omega$. ▶

Infinite Torus Action on Smooth Periodic Functions

We continue the previous exercise on the space of Fourier coefficients of smooth periodic functions, equipped with the functional diffeology, by considering the infinite product T^∞ of tori $U(1)$ as a group of diffeomorphisms.

We denote by T^∞ the set of infinite sequences $\tau = (\tau_n)_{n\in Z}$, where $\tau_n \in T = U(1)$, that is, $\tau_n \in \mathbf{C}$ and $\bar{\tau}_n \tau_n = 1$:

$$T^\infty = \prod_{n\in Z} U(1).$$

Let $\mathcal{E}$ be the set of Fourier coefficients of smooth 1-periodic functions equipped with the pushforward of the functional diffeology, as it is described in “Functional Diffeology on Fourier Coefficients”. There is a natural linear action of T^∞ on $\mathcal{E}$, defined by

$$\tau \cdot Z = (\tau_n)_{n\in Z} \cdot (Z_n)_{n\in Z} = (\tau_n Z_n)_{n\in Z}.$$

Indeed, multiplying by a number of modulus 1 transforms a rapidly decreasing sequence of complex numbers into another, and every $\tau = (\tau_n)_{n\in Z} \in T^\infty$ is invertible

$$(\tau_n)^{-1}_{n\in Z} = (\bar{\tau}_n)_{n\in Z}.$$

Moreover, for every plot $r \mapsto (Z_n(r))_{n\in Z}$ in $\mathcal{E}$, for all $p \in N$,

$$\left|\frac{\partial^p \tau_n Z_n(r)}{\partial r^p}\right| = \left|\frac{\partial^p Z_n(r)}{\partial r^p}\right|.$$

The same holds for the inverse. Hence, the action of $(\tau_n)_{n\in Z}$ is smooth as well as its inverse, thus $(\tau_n)_{n\in Z}$ acts by diffeomorphism. We have then a monomorphism

$$\eta : T^\infty \to GL^\infty(\mathcal{E}).$$

228. Exercise I. We consider the group T^∞ of diffeomorphisms of $\mathcal{E}$. We shall say that a parametrization $r \mapsto \tau(r) = (\tau_n(r))_{n\in Z}$ in T^∞ is *tempered* if the τ_n are smooth and if for every $k \in N$, for every r_0 in the domain of the parametrization, there exists a ball $\mathcal{B}$ centered at r_0, a polynomial $P_k(n)$, and an integer N such that

$$\forall r \in \mathcal{B}, \forall n > N, \quad \left| \frac{\partial^k \tau_n(r)}{\partial r^k} \right| \leq P_k(n). \qquad (\spadesuit)$$

Q1. Show that the tempered parametrizations form a group diffeology on T^∞.

Q2. Show that, equipped with the tempered diffeology, the action of the group T^∞ on $\mathcal{E}$ is smooth.

Q3. Show that for all $N \in N$, the injection $\iota_N : T^N \to T^\infty$ defined by

$$\iota_N(\zeta) = \tau \quad \text{with} \quad \begin{cases} \tau_n = \zeta_n & \text{if } n \in \{1, \dots, N\}, \\ \tau_n = 1 & \text{otherwise,} \end{cases}$$

is an induction.

☞ Proof. For Question 1. Let us show that the condition $(\spadesuit)$ defines a diffeology, actually a sub-diffeology of the product diffeology on the infinite product of tori $T^\infty = \prod_{n\in Z} T$. We denote indifferently U(1) or T, for Torus.

(Covering axiom) If $\tau_n(r) = \tau_n$ is constant in r, for all n, then $|\partial^k \tau_n(r)/\partial r^k|$ is equal to 1 for $k = 0$ and equal to 0 for $k > 0$. It satisfies the condition $(\spadesuit)$.

(Locality axiom) By definition the condition $(\spadesuit)$ is local in the variable r.

(Smooth compatibility axiom) Let $\zeta : (r \mapsto \tau_n(r))_{n\in Z}$ satisfying $(\spadesuit)$ and $F : s \mapsto r$ be a smooth parametrization in the domain of ζ. We have, for all $k > 0$,

$$\frac{\partial^k \tau_n(s)}{\partial s^k} = \sum_{\ell=1}^{k} \frac{\partial^\ell \tau_n(r)}{\partial r^\ell} \cdot Q_{k,\ell}\left(\frac{\partial r}{\partial s}, \ldots, \frac{\partial^k r}{\partial s^k}\right),$$

where the $Q_{k,\ell}$ are polynomials. Therefore,

$$\left|\frac{\partial^k \tau_n(s)}{\partial s^k}\right| \leq \sum_{\ell=1}^{k} \left|\frac{\partial^\ell \tau_n(r)}{\partial r^\ell}\right| \left|Q_{k,\ell}\left(\frac{\partial r}{\partial s}, \ldots, \frac{\partial^k r}{\partial s^k}\right)\right|.$$

Since the function $s \mapsto r$ is smooth, these polynomials are locally absolutely bounded, let

$$M_{k,\ell} = \sup_{r \in \mathcal{B}} \left|Q_{k,\ell}\left(\frac{\partial r}{\partial s}, \ldots, \frac{\partial^k r}{\partial s^k}\right)\right|.$$

Next, let N_ℓ satisfying (♠) for $k = \ell$, and $N' = \sup_{\ell=1\ldots k} N_\ell$, then for all $n > N'$,

$$\left|\frac{\partial^k \tau_n(s)}{\partial s^k}\right| \leq \sum_{\ell=1}^{k} M_{k,\ell} \left|\frac{\partial^\ell \tau_n(r)}{\partial r^\ell}\right| \leq \sum_{\ell=1}^{k} M_{k,\ell} P_\ell(n).$$

Hence, the partial derivatives of the τ_n with respect to r are still dominated by polynomials, and the condition (♠) defines a diffeology on the set T^∞.

For Question 2. Now, let us check that (♠) defines a group diffeology. First of all, let $\tau : r \mapsto (\tau_n(r))_{n\in Z}$ and $\tau' : r \mapsto (\tau'_n(r))_{n\in Z}$ be two plots of T^∞, the derivatives of the terms of the product $\tau_n(r)\tau'_n(r)$ write

$$\frac{\partial^k \tau_n(r)\tau'_n(r)}{\partial r^k} = \sum_{\ell=1}^{k} \binom{k}{\ell} \frac{\partial^{k-\ell} \tau_n(r)}{\partial r^{k-\ell}} \cdot \frac{\partial^\ell \tau'_n(r)}{\partial r^\ell}.$$

Hence,

$$\begin{aligned}\left|\frac{\partial^k \tau_n(r)\tau'_n(r)}{\partial r^k}\right| &\leq \sum_{\ell=1}^{k} \binom{k}{\ell} \left|\frac{\partial^{k-\ell} \tau_n(r)}{\partial r^{k-\ell}}\right| \left|\frac{\partial^\ell \tau'_n(r)}{\partial r^\ell}\right| \\ &\leq \sum_{\ell=1}^{k} \binom{k}{\ell} P_{k-\ell}(n)\, P'_\ell(n).\end{aligned}$$

The partial derivatives of the product $\tau_n\tau'_n$ are still dominated by a polynomial. The inverse mapping is given by $\iota : (\tau_n)_{n\in \mathrm{Z}} \mapsto (\bar{\tau}_n)_{n\in \mathrm{Z}}$, the absolute values of the partial derivatives of an element of the sequence and the corresponding element of the inverse coincide. Therefore, T^∞, equipped with the diffeology (♠) is a diffeological group.

For Question 3. The injection $\iota_\mathrm{N} : \mathrm{T}^\mathrm{N} \to \mathrm{T}^\infty$ is clearly smooth since that, for all plots τ in T^N, the derivatives of the components τ_n of the composite $\iota_\mathrm{N} \circ \tau : r \mapsto (\tau_n(r))_{n\in \mathrm{Z}}$ vanish for n outside $\{1, \ldots, \mathrm{N}\}$, hence

$$n^p \frac{\partial^k \tau_n(r)}{\partial r^k} = 0 \ \text{ if } |n| > \mathrm{N}.$$

Conversely if $\tau : r \mapsto (\tau_n(r))_{n\in \mathrm{Z}}$ is a plot in T^∞ with values in $\iota_\mathrm{N}(\mathrm{T}^\mathrm{N})$, then the components $r \mapsto \tau_n(r)$ are smooth, by definition of the diffeology on T^∞, and hence $\iota_\mathrm{N}^{-1} \circ \tau$ is smooth. Therefore, ι_N is an induction. ▶

229. Exercise II. Let $\alpha = (\alpha_n)_{n\in \mathrm{Z}}$ be a sequence of positive irrational numbers independent over $\mathbf{Q}$, that is, for every sequence $(q_n)_{n\in \mathrm{Z}}$, with finite support, of rational numbers $q_n \in \mathbf{Q}$ one has

$$\sum_{n\in \mathrm{Z}} q_n\alpha_n = 0 \quad \Rightarrow \quad q_n = 0 \ \text{ for all } \ n.$$

Q1. Give an example of such a family of irrational numbers.

Q2. Show that $\iota : \mathbf{R} \mapsto \mathrm{T}^\infty$, defined by

$$\iota(t) = \left(e^{2i\pi\alpha_n t}\right)_{n\in \mathrm{Z}},$$

is an induction.

Proof. For Question 1. Consider a transcendent number α (we assume $\alpha < 1$ even if it is not necessary). We define

$$\alpha_0 = 1 \ \text{ and } \ \alpha_m = \alpha^m - \left[\alpha^m\right] \quad \text{for } m \neq 0,$$

where the bracket denotes the integral part. That defines α_n for all $n \in \mathbf{Z}$. Now, for a finitely supported sequence of rational numbers

q_n, we have

$$\begin{aligned}\sum_{n\in \mathbf{Z}} q_n\alpha_n &= q_0 + \sum_{\substack{n\in \mathbf{Z}\\ n\neq 0}} q_n\left(\alpha^n - [\alpha^n]\right)\\ &= q_0 - \sum_{\substack{n\in \mathbf{Z}\\ n\neq 0}} q_n[\alpha^n] + \sum_{\substack{n\in \mathbf{Z}\\ n\neq 0}} q_n\alpha^n,\end{aligned}$$

with only a finite part of this sum is not zero. Then, if $\sum_{n\in \mathbf{Z}} q_n\alpha_n = 0$ we multiply both sides by α^ℓ, with ℓ big enough to get an algebraic equation in α, with all powers positive. But we assumed α transcendent, then all the coefficients are 0, what implies $q_n = 0$, for all $n \in \mathbf{Z}$.

For Question 2. We need to check first of all that ι is smooth. The successive derivatives are simply

$$\frac{\partial^k e^{2i\pi\alpha_n t}}{\partial t^k} = (2i\pi\alpha_n)^k e^{2i\pi\alpha_n t}.$$

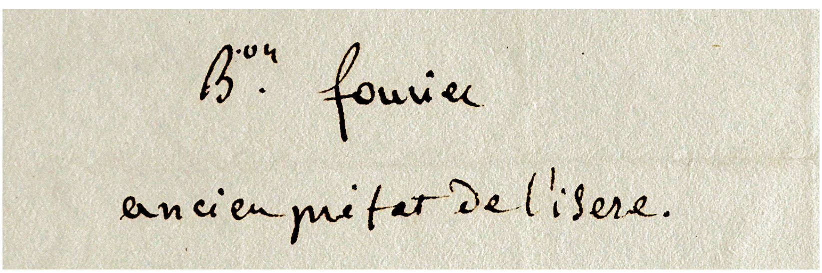
Since $0 < \alpha_n \leq 1$, for all $n \in \mathbf{Z}$, we have

$$\left|\frac{\partial^k e^{2i\pi\alpha_n t}}{\partial t^k}\right| \leq (2\pi)^k.$$

Hence, the derivatives are absolutely dominated by (constant) polynomials and ι is a plot in T^∞. Next, consider a plot in T^∞ with values in $\iota(\mathbf{R})$, the composite with the projection on the first two components gives a plot in T^2 with values in the solenoid

$$(\exp(2i\pi t), \exp(2i\pi\alpha t))_{t\in \mathbf{R}}.$$

But the injection $t \mapsto (\exp(2i\pi t), \exp(2i\pi\alpha t))$ is an induction, see [TB, Exercise 31], therefore, ι is an induction. ▶



Basic 1-Forms on Principal Fiber Bundles

We explicit a criterion for basic 1-forms on general principal bundles.

This is a specialization of the general criterion of paragraph 6.38 in the textbook, for basic 1-forms on principal fiber bundles [**TB**, § 8.11].

230. Exercise. Let $\pi : X \to B$ be a principal fibration with structure group G. Let α be a 1-form on X. Prove that α is *basic*, that is, is the pullback $\pi^*(\beta)$ of a 1-form on B, if and only if:

(1) α vanishes on "vertical plots", that is, the preimages by π of the points of B. In short, $\alpha \upharpoonright \pi^{-1}(b) = 0$, for all $b \in B$.
(2) α is invariant by the structure group, that is, $g_X^*(\alpha) = \alpha$, for all $g \in G$, where g_X denotes the action of g on X.

Give a simple example showing that this criterion is then no more valid even for a 2-form.

☞ Proof. Assume that $\alpha = \pi^*(\beta)$. Let $P : U \to X$ be a vertical plot, then $\alpha(P) = [\pi^*(\beta)](P) = \beta(\pi \circ P)$, but $\pi \circ P = \text{cst}$, thus $\beta(\pi \circ P) = 0$, that is, $\alpha(P) = 0$, see [**TB**, Exercise 96]. Now, let $g \in G$, then $g_X^*(\alpha) = g_X^*(\pi^*(\beta)) = (\pi \circ g_X)^*(\beta)$, but $\pi \circ g_X = \pi$, thus $g_X^*(\alpha) = \alpha$. Therefore, the two conditions above are satisfied.

Conversely, assume that the two conditions above are satisfied, and let $P : U \to X$ and $P' : U \to X$ be two plots such that $\pi \circ P = \pi \circ P'$. Since π is a principal diffeological fibration, there exists, around every point of U, a subdomain $V \subset U$ and a plot $r \mapsto \gamma(r)$ in G, defined on V, such that $P'(r) = \gamma(r)_X(P(r))$, for all $r \in V$ [**TB**, 8.13, Note 1]. Hence,

$$\alpha(P') = \alpha[r \mapsto \gamma(r)_X(P(r))].$$

Now we need the following formula, which is a generalization of [**TB**, 8.37 (♣)]. Let $R(x) : G \to X$ be the orbit map $R(x)(g) = g_X(x)$, let γ be a plot in G defined on some domain U and let P be a plot in X defined on some domain V, then

$$\alpha[(r,s) \mapsto \gamma(r)_X(P(s))]_{\binom{r}{s}} \begin{pmatrix} \delta r \\ \delta s \end{pmatrix} = \alpha[(r,s) \mapsto \gamma(r)_X(P(s))]_{\binom{r}{s}} \begin{pmatrix} \delta r \\ 0 \end{pmatrix}$$

$$+ \alpha[(r,s) \mapsto \gamma(r)_X(P(s))]_{\binom{r}{s}} \begin{pmatrix} 0 \\ \delta s \end{pmatrix} = \alpha[r \mapsto \gamma(r)_X(P(s))]_r(\delta r)$$

$$+\alpha[s \mapsto \gamma(r)_X(P(s))]_s(\delta s) = [R(P(s))^*(\alpha)](\gamma)_r(\delta r) + [\gamma(r)_X^*(\alpha)](P)_s(\delta s).$$

Now, since $R(P(s))$ is the orbit map of the point $x = P(s)$,

$$[R(P(s))^*(\alpha)](\gamma) = [R(P(s))^*(\alpha \upharpoonright \pi^{-1}(b))](\gamma),$$

with $b = \pi(x)$. But, according to the hypothesis, $\alpha \upharpoonright \pi^{-1}(b) = 0$, thus

$$[R(P(s))^*(\alpha)](\gamma)_r(\delta r) = 0.$$

Next, since, by assumption, α is invariant by the action of G, $[\gamma(r)_X^*(\alpha)](P) = \alpha(P)$. Therefore,

$$\alpha[(r,s) \mapsto \gamma(r)_X(P(s))]_{\binom{r}{s}} \begin{pmatrix} \delta r \\ \delta s \end{pmatrix} = \alpha(P)_r(\delta r).$$

This implies in particular $\alpha(P') = \alpha[r \mapsto \gamma(r)_X(P(r))] = \alpha(P)$. Hence, according to the criterion [**TB**, 6.38], there exists a 1-form β on B such that $\alpha = \pi^*(\beta)$. For the second question, consider the 2-form $\omega = dx \wedge dy$ on the plane $\mathbf{R}^2$. We can consider the first projection $\mathrm{pr}_1 : (x,y) \mapsto x$. It is a principal fibration with group $(\mathbf{R}, +)$ for the action $t_{\mathbf{R}^2}(x,y) = (x, y+t)$. The 2-form ω is, in particular, invariant by this action. Moreover, the restriction of ω on a fiber of pr_1 is a linear 2-form on $\mathbf{R}$, thus vanishes. This example satisfies the two conditions above, but since $\omega \neq 0$, ω is not the pullback by pr_1 of a 2-form on $\mathbf{R}$ (which is necessarily 0). ▶

Differential of Holonomy for Torus Bundles

We explicit the holonomy function on a torus bundle on a diffeological space. Here the word torus denotes any quotient $T = \mathbf{R}/\Gamma$, where Γ is a strict subgroup of $\mathbf{R}$. Then, we explicit the differential of the holonomy in terms of the Chain-Homotopy Operator and the curvature.

In this note we will consider a principal fiber bundle $\pi : Y \to X$, with X and Y two diffeological spaces, and with structure group a *torus* $T = \mathbf{R}/\Gamma$, where Γ is any strict subgroup of $\mathbf{R}$. As we know, if $\Gamma = a\mathbf{Z}$ then the torus T is a manifold isomorphic to the circle S^1, and if Γ has more than 1 generator, T is said to be *irrational*, and it's not anymore a manifold. We assume that there exists on Y a connection form λ, that is, a differential 1-from satisfying the two conditions

(1) λ is **invariant** by the action of T:

$$\text{For all } \tau \in T, \quad \tau_Y^*(\lambda) = \lambda,$$

where τ_Y denotes the action of τ on Y.

(2) λ is **calibrated**:

$$\text{For all } y \in Y, \quad \hat{y}^*(\lambda) = \theta,$$

where $\hat{y} : T \to Y$ is the orbit map $\hat{y}(\tau) = \tau_Y(y)$, and θ is the canonical 1-form on T, pushforward of the canonical 1-form dt on $\mathbf{R}$.

Then, the connection is defined by the *horizontal paths* in Y, and they are

$$\mathrm{Hor}(Y) = \{\gamma \in \mathrm{Paths}(Y) \mid \lambda(\gamma)_t = 0 \text{ for all } t \in \mathbf{R}\}.$$

See [TB] for the details of these constructions, in particular § 8.37.

58. Loops bundles

<u>231. Exercise I.</u> Show that the pushforward

$$\pi_* : \mathrm{Loops}(Y) \to \mathrm{Loops}(X), \text{ defined by } \pi_*(\tilde{\ell}) = \pi \circ \tilde{\ell},$$

is surjective.

☞ Proof. Consider a loop ℓ in X, that is, $\ell \in \mathcal{C}^\infty(\mathbf{R}, X)$ and $\ell(0) = \ell(1)$. The pullback of Y by ℓ is a principal bundle on $\mathbf{R}$ with a connection, therefore it is trivial [TB, § 8.34]: there exists an equivariant diffeomorphism $\phi : \mathbf{R} \times T \to \ell^*(Y)$. Let $\phi(t,\tau) = (t, f(t,\tau))$, with $f(t,\tau) \in Y_{\ell(t)}$.

$$\begin{array}{ccccc} \mathbf{R}\times T & \xrightarrow{\phi} & \ell^*(Y) & \xrightarrow{\mathrm{pr}_2} & Y \\ & \searrow^{\mathrm{pr}_1} & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \pi} \\ & & \mathbf{R} & \xrightarrow[\ell]{} & X \end{array}$$

Then, by equivariance, $\phi(t,\tau) = (t, \tau_Y(f(t)))$, where $f \in \mathrm{Paths}(Y)$ and $\pi \circ f = \ell$. There exists a unique $a \in T$ such that $f(1) = a_Y(f(0))$. Now, since T is connected there exists $\tau \in \mathrm{Paths}(T)$ such that $\tau(0) = 1$ and $\tau(1) = a^{-1}$. Then, $\tilde{\ell} : t \mapsto \tau(t)_Y(f(t))$ is a path in Y such that $\pi \circ \tilde{\ell} = \ell$, and $\tilde{\ell}(0) = \tilde{\ell}(1) = f(0)$. Therefore, $\tilde{\ell} \in \mathrm{Paths}(Y)$ and $\pi \circ \tilde{\ell} = \ell$. ▶

Now, there is more than π_* being surjective. Using the same ideas than in exercise I:

<u>232. Exercise II.</u> Show that the pushforward $\pi_* : \mathrm{Loops}(Y) \to \mathrm{Loops}(X)$ is not only surjective but even a subduction.

☞ Proof. Let $r \mapsto \ell_r$ be a plot in Loops(X) defined on a small open ball B centered at 0 in some $\mathbf{R}^n$. Since $r \mapsto \ell_r$ is smooth, the map

$Q : (r, t) \mapsto \ell_r(t)$ defined on $B \times \mathbf{R}$ is a plot of X. Since we have a connection on $\pi : Y \to X$, we have a connection, by pullback, on $\mathrm{pr}_1 : Q^*(Y) \to B \times \mathbf{R}$. And since $B \times \mathbf{R}$ is contractible, the fibration $\mathrm{pr}_1 : Q^*(Y) \to B \times \mathbf{R}$ is trivial, that is, there exists an equivariant diffeomorphism $\phi : B \times \mathbf{R} \times T \to Q^*(Y)$. By equivariance, $\phi(r, t, \tau) = (r, t, \tau_Y(f_r(t)))$, where $(r, t) \mapsto f_r(t)$ is a plot of Y. That is equivalent to say that $r \mapsto f_r$ is a plot of Paths(Y). We have then $\pi \circ f_r(t) = \ell_r(t)$.

$$\begin{array}{ccccc} B \times \mathbf{R} \times T & \xrightarrow{\ \phi\ } & \ell^*(Y) & \xrightarrow{\ \mathrm{pr}_3\ } & Y \\ & \searrow^{\mathrm{pr}_{1,2}} & \downarrow \mathrm{pr}_1 & & \downarrow \pi \\ & & B \times \mathbf{R} & \xrightarrow[\ \ell\ ]{} & X \end{array}$$

Thus, there exists $r \mapsto a_r \in T$ such that $f_r(1) = (a_r)_Y(f_r(0)$. Let $\alpha : r \mapsto f_r(0)$ and $\beta : r \mapsto f_r(1)$. Since α and β are homotopic, the pullbacks $\alpha^*(Y)$ and $\beta^*(Y)$ are equivalent (actually trivial) [**TB**, § 8.34]. Therefore, $r \mapsto a_r$ is a plot of T, as well as $r \mapsto a_r^{-1}$. Now, the projection class : $\mathbf{R} \to T$ is the universal covering. The ball B being contractible, there exists a unique lifting $r \mapsto \bar{a}_r$ of $r \mapsto a_r^{-1}$ such that $\bar{a}_0 = 0$ [**TB**, § 8.25]. Consider then $(r, t) \mapsto \tau_r(t) = \mathrm{class}(t\bar{a}_r)$. The parametrization $r \mapsto \tau_r$ is a plot of Paths(T) such that $\bar{a}_r(0) = 1$ and $\bar{a}_r(1) = a_r^{-1}$. Then, let $\tilde{\ell}_r(t) = \tau_r(t)_Y(f_r(t))$, it is a plot of Y such that $\tilde{\ell}_r(0) = f_r(0)$ and $\tilde{\ell}_r(1) = \tau_r(1)_Y(f_r(1)) = (a_r^{-1})_Y((a_r)_Y(f_r(0)) = f_r(0)$. Thus, $r \mapsto \tilde{\ell}_r$ is a plot of Loops(Y), and a lifting of $r \mapsto \ell_r$, that is, such that $\pi \circ \tilde{\ell}_r = \ell_r$, for all $r \in B$. Hence, every plot in Loops(X) has a local smooth lifting in Loops(Y) everywhere. Therefore, $\pi_* :$ Loops(Y) $\to$ Loops(X) is a subduction. ▶

59. Holonomy function

In this section we shall explicit the holonomy H of the connection defined by λ, and compute its differential.

233. Exercise III. Show that there exists a smooth map

$$H : \mathrm{Loops}(X) \to T \quad \text{defined by} \quad H(\ell) = \mathrm{class}\left(\int_{\tilde{\ell}} \lambda\right),$$

for all $\tilde{\ell} \in \mathrm{Loops}(Y)$ such that $\pi \circ \tilde{\ell} = \ell$. And where class denotes the canonical projection from $\mathbf{R}$ to T.

☞ Proof. Let $\tilde{\ell}$ and $\tilde{\ell}'$ be two loops in Y projecting on ℓ. Since $\pi : Y \to X$ is a diffeological fiber bundle, there exists a loop τ in T such that $\tilde{\ell}'(t) = \tau(t)_Y(\tilde{\ell}(t))$. According to [TB, § 8.37], we have

$$\lambda\big(t \mapsto \tau(t)_Y(\tilde{\ell}(t))\big)_t(1) = \tau^*(\theta)_t(1) + \lambda(\tilde{\ell})_t(1).$$

Then,

$$\begin{aligned}
\int_{\tilde{\ell}'} \lambda &= \int_0^1 \lambda(\tilde{\ell}')_t(1)\ dt \\
&= \int_0^1 \lambda(t \mapsto \tau(t)_Y(\tilde{\ell}(t)))_t(1)\ dt \\
&= \int_0^1 \tau^*(\theta)_t(1)\ dt + \int_0^1 \lambda(t \mapsto \tilde{\ell}(t))_t(1)\ dt \\
&= \int_\tau \theta + \int_{\tilde{\ell}} \lambda.
\end{aligned}$$

But since τ is a loop in T and θ is the canonical 1-form on $T = \mathbf{R}/\Gamma$, $\int_\tau \theta \in \Gamma$. Therefore, $\mathrm{class}\left(\int_{\tilde{\ell}'} \lambda\right) = \mathrm{class}\left(\int_{\tilde{\ell}} \lambda\right)$. We get the map H, which is well defined: $H(\ell) = \mathrm{class}\left(\int_{\tilde{\ell}} \lambda\right)$. Let us denote $\tilde{H}$ the integral of λ on the loops in Y. We have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Loops}(Y) & \xrightarrow{\tilde{H}} & \mathbf{R} \\
{\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle \mathrm{class}} \\
\mathrm{Loops}(X) & \xrightarrow[H]{} & T
\end{array}$$

Now, since π_* is a subduction, according to the previous exercise, the map H is smooth. ▶

The values of H is exactly the group of holonomy of the connection defined by λ, see [TB, § 8.35]. It is a subgroup of T, it is either discrete or the whole T. A way to check if it is discrete is to compute its "differential". We will define it as:

$$d_T H = H^*(\theta).$$

234. Exercise IV. Show that

$$d_T H + \mathcal{K}\omega = 0,$$

where $\mathcal{K}$ is the Chain-Homotopy Operator [**TB**, § 6.83].

☞ Proof. Remark that

$$\tilde{H} = \left[\tilde{\ell} \mapsto \int_{\tilde{\ell}} \lambda\right] = \mathcal{K}\lambda.$$

Then, apply the fundamental property of the Chain-Homotopy Operator, restricted to the space of loops of Y. That gives

$$d(\mathcal{K}\lambda) + \mathcal{K}(d\lambda) = [(\hat{1}^* - \hat{0}^*) \restriction \mathrm{Loops}(Y)](\lambda) = 0.$$

Recalling that $d\lambda = \pi^*(\omega)$, we get

$$d\tilde{H} + \mathcal{K}(\pi^*(\omega)) = 0.$$

The variance of the Chain-Homotopy Operator states that the following diagram is commutative, see [**TB**, § 6.84].

$$\begin{array}{ccc} \Omega^k(X) & \xrightarrow{\mathcal{K}_X} & \Omega^{k-1}(\mathrm{Paths}(X)) \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle (\pi_*)^*} \\ \Omega^k(Y) & \xrightarrow{\mathcal{K}_Y} & \Omega^{k-1}(\mathrm{Paths}(Y)) \end{array}$$

Thus (forgetting the indices on $\mathcal{K}$),

$$d\tilde{H} + (\pi_*)^*(\mathcal{K}\omega) = 0.$$

But $d\tilde{H} = \tilde{H}^*(dt)$, and $dt = \mathrm{class}^*(\theta)$, thus $d\tilde{H} = \tilde{H}^*(\mathrm{class}^*(\theta)) = (\mathrm{class} \circ \tilde{H})^*(\theta) = (H \circ \pi_*)^*(\theta) = (\pi_*)^*(d_T H)$. Hence,

$$(\pi_*)^*(d_T H) + (\pi_*)^*(\mathcal{K}\omega) = (\pi_*)^*(d_T H + \mathcal{K}\omega) = 0.$$

Now, since π_* is a subduction, according to the previous exercise, and thanks to [**TB**, § 6.39],

$$d_T H + \mathcal{K}\omega = 0.$$

And that is the expression of the differential of the holonomy. We get from this identity that if the curvature ω vanishes, then the holonomy is discrete and the fiber bundle reduces to a covering. ▶

Conclusion By definition [**TB**, § 6.83], the Chain-Homotopy Operator is defined by

$$\mathcal{K}\omega = i_\tau \circ \Phi(\omega),$$

where $\tau \in \mathrm{Hom}^\infty(\mathbf{R}, \mathrm{Diff}(\mathrm{Loops}(X)))$ is the action of reparametrization of paths:

$$\tau(\varepsilon)(\gamma) = [t \mapsto \gamma(t+\varepsilon)],$$

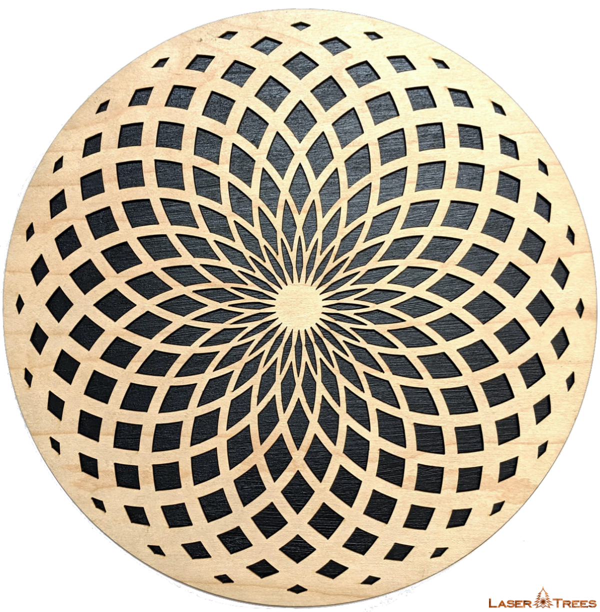
and $\Phi(\omega)$ is the mean value of the "time-pullbacks":

$$\Phi(\omega) = \int_0^1 \hat{t}^*(\omega)\ dt,$$

with $\hat{t}(\gamma) = \gamma(t)$. That way, the differential of the holonomy writes:

$$d_{\mathrm{T}}\mathrm{H} + i_\tau \mathrm{F} = 0 \quad \text{with} \quad \mathrm{F} = \Phi(\omega).$$

That is the (infinitesimal) equivariant cohomology way of expressing this differential: H is the moment map associated with the reparametrization group action on Loops(X), with respect to the 2-form F.



Take a good look, this torus flat!

Non-Symplectic Manifold with Injective Universal Moment Map

This addendum is an exercise, with a detailed solution, made with the Note 2 of the article 9.23 of the textbook. This example shows how the condition of transitivity of the automorphisms is necessary for being a symplectic manifold.

Considering the program of symplectic diffeology I have suggested to call symplectic differential form on a diffeological space X any closed 2-form ω satisfying the two following properties:

(1) The universal moment map $\mu_\omega : X \to \mathcal{G}_\omega$ is injective.
(2) The local automorphisms $\mathrm{Diff}_{\mathrm{loc}}(X, \omega)$ are transitive on X.

The first condition is to avoid non zero characteristics of ω because of the theorem [**TB**, § 9.26] claiming that for manifolds homogeneous under $\mathrm{Diff}(X, \omega)$, the characteristics of ω are precisely the pre-images of the universal moment map.

The second condition is the transposition to diffeology of Darboux theorem satisfied by symplectic manifold. It could be strengthened by considering the group of automorphisms, which is still satified for manifolds, but it doesn't seem necessary so be it.

Now, we shall see why the second condition is necessary by constructing an example of a closed 2-form on a manifold for which the univesal moment map is injective but the group of automorphism is not transitive.

<u>235. Exercise.</u> Let us consider the real plane $\mathbf{R}^2$ equipped with the 2-form

$$\omega = (x^2 + y^2)\, dx \wedge dy.$$

Q1. Why is ω closed?

Q2. Describe the kernel of ω. We admit that the group of compact supported automorphisms of a symplectic manifold is transitive, deduce that $\mathrm{Diff}(\mathbf{R}^2, \omega)$, the group of automorphisms of ω, has 2 orbits: $\{0_{\mathbf{R}^2}\}$ and $\mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$.

Q3. Tell why every automorphism of $(\mathbf{R}^2, \omega)$ is Hamiltonian.

Q4. Exhibit the unique equivariant universal moment map μ_ω for $(\mathbf{R}^2, \omega)$ such that $\mu_\omega(0_{\mathbf{R}^2}) = 0_{\mathcal{G}^*}$. Why is it unique?

Q5. Show that if $z = (x, y) \neq (0, 0)$, then $\mu_\omega(z) \neq 0_{\mathcal{G}^*}$. Conclude that μ_ω is injective.

Proof. Q1. ω is closed because it is a 2-form on a 2-dimensional space (op. cit. § 6.39).

Q2. Let $z = (x, y) \in \mathbf{R}^2$, by definition

$$\ker(\omega_z) = \{u \in \mathbf{R}^2 \mid \omega_z(u)(v) = 0 \text{ for all } v \in \mathbf{R}^2\}.$$

But, $\omega_z(u)(v) = (x^2 + y^2)\det(u\, v)$. Thus, since $\det(\cdot\,\cdot)$ is nondegenerate, if $z \neq (0, 0)$ then $\ker(\omega_z) = \{0_{\mathbf{R}^2}\}$, else $\ker(\omega_0) = \mathbf{R}^2$. It follows that, since an automorphism of ω induces an isomorphism on the kernel, the origin $0_{\mathbf{R}^2}$ must be mapped onto itself by any automorphism of ω. Indeed, $0_{\mathbf{R}^2}$ is the only point with kernel $\mathbf{R}^2$. Then, for all $\varphi \in \mathrm{Diff}(\mathbf{R}^2, \omega)$, $\mathbf{R}^2 - \{0\}$ is invariant by φ, thus $\varphi \restriction \mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$ is a symplectomorphism of $\mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$. Now, $\mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$ is open and $\omega \restriction \mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$ is symplectic, according to the assumption: for every two points z and z' in $\mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$ there exists a compact supported automorphism φ of $(\mathbf{R}^2 - \{0_{\mathbf{R}^2}\}, \omega \restriction \mathbf{R}^2 - \{0_{\mathbf{R}^2}\})$ mapping z to z'. But since a compact in $\mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$ is a closed and bounded subset, the complement of the support contains an open neighborhood of $0_{\mathbf{R}^2}$ on which φ is the identity, thus φ can be smoothly extended by $\varphi(0_{\mathbf{R}^2}) = 0_{\mathbf{R}^2}$. This extension, still mapping z to z', satisfies obiously $\varphi^*(\omega) = \omega$ on $\mathbf{R}^2$ and then belongs to $\mathrm{Diff}(\mathbf{R}^2, \omega)$. Therefore, $\mathrm{Diff}(\mathbf{R}^2, \omega)$ is transitive on $\mathbf{R}^2 - \{0\}$, that is, the group $\mathrm{Diff}(\mathbf{R}^2, \omega)$ has two orbits in $\mathbf{R}^2$: $\{0_{\mathbf{R}^2}\}$ and $\mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$.

Q3. Since $\mathbf{R}^2$ is contractible (precisely, has a vanishing first homology), the automorphisms $\mathrm{Diff}(\mathbf{R}^2, \omega)$ are Hamiltonian (op. cit. § 9.7 and 9.15).

Q4. Since the action of $\mathrm{Diff}(\mathbf{R}^2, \omega)$ has a fixed point, $0_{\mathbf{R}^2}$, the moment map is exact and there exists an invariant primitive μ_ω (op. cit. § 9.10, Note 2). Actually, applying the expressions (♦) and (♥) of (op. cit. § 9.20), to a path p, connecting $0_{\mathbf{R}^2}$ to $z = (x, y)$, an equivariant primitive μ_ω (op. cit. § 9.9) of the 2-points moment map(op. cit. § 9.8) is given, for every plot F of $\mathrm{Diff}(\mathbf{R}^2, \omega)$, by

$$\mu_\omega(z)(\mathrm{F})_r(\delta r) = \int_0^1 \omega_{p(t)}(\dot{p}(t))(\delta p(t))\, dt,$$

with

$$\delta p(t) = [\mathrm{D}(\mathrm{F}(r))(p(t))]^{-1} \frac{\partial \mathrm{F}(r)(p(t))}{\partial r}(\delta r).$$

This moment is unique because two moment maps differ only by a constant in $\mathcal{G}^*$ ($\mathbf{R}^2$ is connected) (op. cit. § 9.9), and the constant is fixed by $\mu_\omega(0_{\mathbf{R}^2}) = 0_{\mathcal{G}^*}$.

Q5. The proof that the moment map μ_ω restricted to $\mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$ is injective is contained in the proof of (op. cit. § 9.23, B) by chosing a real smooth function f on $\mathbf{R}^2$ vanishing outside a small ball centered at $z' \neq z$ not containing z nor $0_{\mathbf{R}^2}$. Now we have just to show that for all $z \in \mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$, $\mu_\omega(z) \neq 0_{\mathcal{G}^*}$. For that we will apply the previous formula to

$$p(t) = tz \quad \text{and} \quad \mathrm{F}(r) = \begin{pmatrix} \cos(2\pi r) & -\sin(2\pi r) \\ \sin(2\pi r) & cos(2\pi r) \end{pmatrix}$$

with

$$t \in \mathbf{R}, \quad z = (x, y) \in \mathbf{R}^2, \quad r \in \mathbf{R} \ \text{ and } \ \delta r = 1.$$

By linearity, $\mathrm{D}(\mathrm{F}(r))(p(t)) = \mathrm{F}(r)$, and then

$$\begin{aligned}\delta p(t) &= \mathrm{F}(r)^{-1}\frac{\partial \mathrm{F}(r)}{\partial r}(p(t)) \\ &= \begin{pmatrix} \cos(2\pi r) & \sin(2\pi r) \\ -\sin(2\pi r) & cos(2\pi r)\end{pmatrix} \times 2\pi \times \begin{pmatrix} -\sin(2\pi r) & -\cos(2\pi r) \\ \cos(2\pi r) & -\sin(2\pi r)\end{pmatrix}\begin{pmatrix} tx \\ ty \end{pmatrix} \\ &= 2\pi \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}\begin{pmatrix} tx \\ ty \end{pmatrix} = 2\pi \begin{pmatrix} -ty \\ tx \end{pmatrix}.\end{aligned}$$

On the other hand, $\dot{p}(t) = \frac{d(tz)}{dt} = z$. Hence,

$$\dot{p}(t) = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{and} \quad \delta p(t) = 2\pi \begin{pmatrix} -ty \\ tx \end{pmatrix}.$$

Therefore,

$$\begin{aligned}\mu_\omega(z)(\mathrm{F})_r(\delta r) &= \int_0^1 \omega_{p(t)}(\dot{p}(t))(\delta p(t))dt = 2\pi \int_0^1 \omega_{tz}\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} -ty \\ tx \end{pmatrix} dt \\ &= 2\pi \int_0^1 ((tx)^2 + (ty)^2)\det\begin{pmatrix} x & -y \\ y & x \end{pmatrix} tdt\end{aligned}$$

$$= 2\pi \int_0^1 (x^2+y^2)(x^2+y^2)t^3 dt = 2\pi(x^2+y^2)^2 \int_0^1 t^3 dt = \frac{2\pi}{4}(x^2+y^2)^2.$$

Hence, if $z = (x, y) \neq (0, 0)$ the value of the moment map above, computed on the 1-path F, is not zero, which implies $\mu_\omega(z) \neq 0_{\mathcal{G}^*}$.

In conclusion, $\mu_\omega(0_{\mathbf{R}^2}) = 0_{\mathcal{G}^*}$, $\mu_\omega \restriction \mathbf{R}^2 - \{0_{\mathbf{R}^2}\}$ is injective and if $z \neq 0_{\mathbf{R}^2}$ then $\mu_\omega(z) \neq 0_{\mathcal{G}^*}$, therefore μ_ω is injective. ▶

So, what is this exercise all about? The paragraph 9.23 states that a closed 2-form ω on a manifold is symplectic, that is, nondegenerate, if and only if its group of automorphisms is transitive and the universal moment map is injective. This exercise shows that the injectivity of the universal moment map is not sufficient (and the condition of transitivity is necessary), since it exhibits a non symplectic closed 2-form on $\mathbf{R}^2$ with an injective universal moment map.

On Riemannian Metric in Diffeology

In this note, we attempt to establish the formal framework of Riemannian diffeology. This involves providing a definition of a Riemannian metric that coincides with the standard definition of manifolds. It's easy to define a symmetric, covariant positive 2-tenor, the tricky part being deciding what we mean by positive definite.

I have opted for a pointwise approach to the definition of a Riemannian metric on diffeological spaces.

60. Pointed or pointwise diffeology

The notion of pointed or pointwise diffeology has been used at a few places already: for the dimension in diffeology [TB, § 2.19/20], or for the construction of differential p-forms bundles and p-vectors bundles (op. cit. § 6.45). We can detach the definition of pointwise diffeological objects from an ambient diffeology, and make it depend solely on a pointed diffeology. First, let us recall that :

236. Pointed parametrization. A *pointed parametrization* in a set X at a point x is any parametrization $\mathrm{P} : \mathrm{U} \to \mathrm{X}$ such that: $0 \in \mathrm{U}$ and $\mathrm{P}(0) = x$. We denote by $\mathrm{Param}_x(\mathrm{X})$ the set of all parametrizations pointed at x.

237. Pointed diffeology. Let X a set and $x \in \mathrm{X}$, we call a *pointed diffeology* at x any set $\mathcal{D}_x \subset \mathrm{Param}_x(\mathrm{X})$ that satisfies the two axioms:

(1) The constant parametrization $0 \mapsto x$ belongs to $\mathcal{D}_x$.
(2) For all parametrization $\mathrm{P} : \mathrm{U} \to \mathrm{X}$ belonging to $\mathcal{D}_x$ and for all smooth parametrization $\mathrm{F} : \mathrm{V} \to \mathrm{U}$, pointed at 0, $\mathrm{P} \circ \mathrm{F}$ belongs to $\mathcal{D}_x$.

Note that a pointed diffeology at x is always the germ at the point x of the diffeology it generates (op. cit. § 1.6). And, for any diffeology $\mathcal{D}$, the subset $\mathcal{D}_x \subset \mathcal{D}$ of plots pointed at x is a pointed diffeology.

238. Pointed path. A pointed path at $x \in \mathrm{X}$ is a pointed smooth parametrization at x, defined on $\mathbf{R}$ or some interval $]a, b[$.

239. Pointed p-form. *A pointed p-form* at x is a map α_x that associates to every pointed plot P at x a linear p-form $\alpha_x(\mathrm{P}) \in \Lambda^p(\mathbf{R}^n)$ at $0 \in \mathrm{dom}(\mathrm{P}) \subset \mathbf{R}^n$, such that $\alpha_x(\mathrm{F} \circ \mathrm{P}) = \mathrm{F}^*(\alpha_x(\mathrm{P}))_0$, that is:

$$\alpha_x(\mathrm{F} \circ \mathrm{P})(v_1, \ldots, v_p) = \alpha_x(\mathrm{P})(\mathrm{M}v_1, \ldots, \mathrm{M}v_p), \quad \text{with } \mathrm{M} = \mathrm{D}(\mathrm{F})(0),$$

for all smooth parametrization F in dom(P) pointed at 0. We shall denote the space of pointed p-forms at x by $\lambda^p_x(\mathrm{X})$.

Note. The value at x of a (global) p-form is a pointed p-form, but maybe not all pointed p-forms are values of a (global) p-form. We have denoted by $\Lambda^p_x(\mathrm{X})$ the space of values of p-forms at x (op. cit. § 6.45). Actually, according to our notations:

$$\Lambda^p_x(\mathrm{X}) \subset \lambda^p_x(\mathrm{X}).$$

61. Smooth covariant tensor

We recall (op. cit. § 6.20 Note, 6.21) that a *smooth covariant tensor* on a diffeological space is a map ε that associates to every plot P in X a smooth covariant tensor $\varepsilon(\mathrm{P})$ on $\mathrm{U} = \mathrm{dom}(\mathrm{P})$, such that

$$\varepsilon(\mathrm{P} \circ \mathrm{F}) = \mathrm{F}^*(\varepsilon(\mathrm{P}))$$

for all smooth parametrization $\mathrm{F} : \mathrm{V} \to \mathrm{U}$. The tensor ε is symetric if $\varepsilon(\mathrm{P})$ is symetric for all P. In the following we deal with symetric

2-tensor and we denotes their space by

$$\Sigma^2(X),$$

and the space of symetric 2-tensor on the Euclidean subset U by $\Sigma^2(U)$ with the identification $\varepsilon \sim \varepsilon(1_U)$.

About the notations, if ε is a smooth covariant k-tensor on a domain $U \subset \mathbf{R}^n$, we denote by $\varepsilon_r(v_1)\cdots(v_k)$ the evaluation of ε at the point $r \in U$, applied to the k-uple of vectors $v_1, \cdots, v_k \in \mathbf{R}^n$. For $n = 1$, a vector is just a number and 1 is the canonical basis vector.

62. Riemannian metric on a diffeological space

240. Riemannian metric. We shall define, for now, a Riemannian metric on a diffeological space X a covariant 2-tensor that satisfies the following conditions:

- (Symetric) The tensor g is symetric:

$$g \in \Sigma^2(X).$$

- (Positive) For all path $\gamma \in \text{Paths}(X)$, $g(\gamma) \geq 0$, that is

$$g(\gamma)_t(1)(1) \geq 0 \quad \text{for all } t \in \mathbf{R}.$$

  Actually we can restrict the case to paths defined on $\mathbf{R}$ or on intervals $]a, b[$, in that case $g(\gamma)_t(1)(1) \geq 0$ for all $t \in \text{dom}(\gamma)$ obviously.
- (Definite) The tensor g is positive definite:

$$g(\gamma)_t(1)(1) = 0 \text{ if and only if } \forall \alpha \in \Omega^1(X),\ \alpha(\gamma)_t(1) = 0.$$

The last condition can be weakened by considering pointed differential forms, as defined above. Considering the space $\lambda^k_x(X)$ of pointed k-forms at x by, the positivity condition becomes:

- (Definite') *The tensor g is positive definite if for all point $x \in X$, for all path γ pointed at x :*

$$g(\gamma)_0(1)(1) = 0 \textit{ if and only if } \forall \alpha_x \in \lambda^1_x(X),\ \alpha_x(\gamma)_0(1) = 0.$$

It is not clear what definition is the best, for many examples built with manifolds and spaces of smooth maps they do coincide. But they may differ in general and, depending on the problem, one must choose one or the other.

241. Length and energy of a path. Let g be a Riemannian metric on a diffeological space X. For all path γ in X, we define its *length* and its *energy* by:

$$\mathrm{Length}(\gamma) = \int_0^1 \sqrt{g(\gamma)_t(1)(1)}\,dt, \text{ and } \mathrm{E}(\gamma) = \frac{1}{2}\int_0^1 g(\gamma)_t(1)(1)\,dt\,.$$

63. How does this fit?

242. Exercise 1. For all $x \in \mathrm{X}$ we say that a path γ is centered at x if $\gamma(0) = x$. Let g be a symmetric 2-tensor on X. Show that:

- g is positive if for all $x \in \mathrm{X}$, for all path γ centered at x, $g(\gamma)_0(1)(1) \geq 0$.
- g is definite if for all $x \in \mathrm{X}$, for all path γ centered at x, $g(\gamma)_0(1)(1) = 0$ implies that for all 1-form α on X, pointed or not, $\alpha(\gamma)_0(1) = 0$.

☞ Proof. It is a part of the definition that if $g(\gamma)_t(1)(1) \geq 0$ for all path γ in X and all $t \in \mathrm{dom}(\gamma)$, then, for all $x \in \mathrm{X}$, for all path γ centered in x, $g(\gamma)_0(1)(1) \geq 0$. Conversely, assume that for all $x \in \mathrm{X}$, for all path γ centered in x, $g(\gamma)_0(1)(1) \geq 0$. Let γ' be a path in X and $t \in \mathrm{dom}(\gamma')$, let $x = \gamma'(t)$. Let $\mathrm{T}_t(t') = t' + t$ be the translation by t in $\mathbf{R}$ and $\gamma = \gamma' \circ \mathrm{T}_t$ Then,

$$\begin{aligned} g(\gamma)_0(1)(1) &= g(\gamma' \circ \mathrm{T}_t)_0(1)(1) \\ &= \mathrm{T}_t^*(g(\gamma'))_0(1)(1) \\ &= g(\gamma')_t(1)(1) \qquad \text{because } \mathrm{D}(\mathrm{T}_t)_0(1) = 1. \end{aligned}$$

Thus, for all $t \in \mathrm{dom}(\gamma')$, $g(\gamma')_t(1)(1) \geq 0$.

The same use of translation by t proves the second proposition. ▶

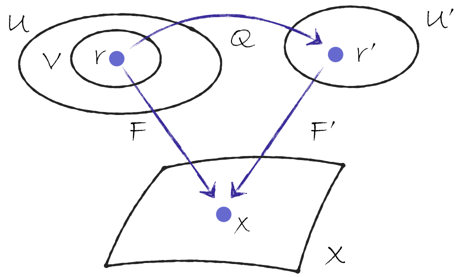


Figure 34. The transition function.

243. Exercise 2. Show that for $X = M$ be a manifold, this definition coincide with the standard definition. We choose the definition of positive definite metric with pointed forms. Say why on manifold the two conditions coincide.

Proof. Let $\mathcal{A}$ be an atlas of M and let $\dim(M) = n$. By definition, for all chart $F \in \mathcal{A}$, $g(F)$ is a symetric 2-tensor on dom(F), since a chart is a particular plot. Let $x \in M$ and two charts $F, F' \in \mathcal{A}$ such that $F(r) = F(r') = x$. Since $\mathcal{A}$ is a generating family, there exists an open neighborhood $V \subset \mathrm{dom}(F)$ of r and a plot $Q : V \to \mathrm{dom}(F')$ such that $F' \circ Q = F \restriction V$, $Q(r) = r'$, and we can choose V such that $Q(V) \subset \mathrm{dom}(F')$. Thus, $g(F \restriction V) = g(F \circ Q) = Q^*(g(F))$, but $Q = F'^{-1} \circ F \restriction V$ is the transition diffeomorphism, hence: $g(F) = (F'^{-1} \circ F)^*(g(F'))$ which is the definition of a 2-tensor on a manifold.

Now, let F be a chart of M. As we said $g(F)$ is a symmetric 2-tensor on $\mathrm{dom}(F) \subset \mathbf{R}^n$. Let $g(F)_r$ be its value in r, and $x = F(r)$. Let $v \in \mathbf{R}^n$ and $\gamma_v : t \mapsto r + tv$, γ_v is a smooth path in dom(F), defined on some interval in $\mathbf{R}$ and centered at r. Let $\gamma^v = F \circ \gamma$, then $\gamma^v(0) = F(r) = x$. Then:

$$\begin{aligned} g(\gamma^v)_0(1)(1) &= g(F \circ \gamma_v)_0(1)(1) \\ &= \gamma_v^*(g(F))_0(1)(1) \\ &= g(F)_{\gamma_v(0)}(\dot\gamma_v(0))(\dot\gamma_v(0)), \text{ with } \dot\gamma_v(t) = \frac{d\gamma_v(t)}{dt} \\ &= g(F)_r(v)(v) \end{aligned}$$

Since $g(\gamma)_0(1)(1) \geq 0$ for all γ, then, for $\gamma = \gamma^v$, $g(F)_r(v)(v) \geq 0$ for all $r \in \text{dom}(F)$ and $v \in \mathbf{R}^n$. Thus, $g(F)$ is a non-negative symmetric 2-tensor.

Now, let F be a chart and let us check that $g(F)$ is positive definite. Let $r \in \text{dom}(F)$ and $v \in \mathbf{R}^n$. Let $x = F(r)$. Assume that $g(F)_r(v, v) = 0$. Let $\gamma_v(t) = r + tv$ and $\gamma^v = F \circ \gamma_v$, then:

$$\begin{aligned} g(F)_r(v)(v) &= g(F)_{\gamma_v(0)}(\dot\gamma_v(0))(\dot\gamma_v(0)) \\ &= \gamma_v^*(g(F))_0(1)(1) \\ &= g(F \circ \gamma_v)_0(1)(1) \\ &= g(\gamma^v)_0(1)(1) \end{aligned}$$

So, if $g(F)_r(v, v) = 0$ then $g(\gamma^v)_0(1)(1) = 0$, which implies, by hypothesis, that for all 1-form α_x pointed at x, $\alpha_x(\gamma^v) = 0$. Consider now the coordinate 1-forms $e_i^* : v \mapsto v^i$, for all $v = \sum_i v^i e_i$, where $(e_i)_{i=1}^n$ is the canonical basis of $\mathbf{R}^n$. Push the form e_i^* onto M by the chart F: Let ε_i^x defined as follow: for all plots $P : U \to X$ pointed at x, there exists a smooth parametrization Q pointed at r, with $F(r) = x$, defined on neighborhood V of $0 \in U$ such that $P \restriction V = F \circ Q$. Then, let $\varepsilon_i^x(P) = Q^*(e_i^*)$, this is a 1-form centered at x. Indeed, for $P' = P \circ F$, $Q' = Q \circ F$ and $\varepsilon_i^x(P \circ F) = (Q \circ F)^*(e_1^*) = F^*(Q^*(e_i^*)) = F^*(\varepsilon_i^x(P))$. Now, since g is assumed to be positive definite: $\varepsilon_i^x(\gamma^v) = 0$, but $\gamma^v = F \circ \gamma_v$, thus $\varepsilon_i^x(\gamma^v) = \gamma_v^*(e_i^*) = (e_i^*)_r(\dot\gamma_v(0)) = e_i^*(v) = v^i$.

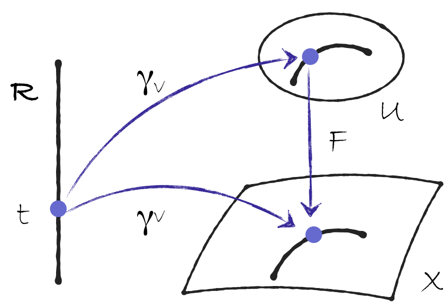


Figure 35. A path in a chart.

Hence, for all i, $v_i = 0$ and then $v = 0$. The 2-tensor $g(\mathrm{F})$ defined on $\mathrm{dom}(\mathrm{F})$ is a positive definite metric. ▶

__244. Exercise 3.__ For two vectors $v, v' \in \mathbf{R}^3$, denote by $\langle v, v' \rangle$ their ordinary scalar product. Let $\gamma \in \mathrm{Paths}(\mathbf{R}^3)$, call a *variation* of γ a path $t \mapsto (x_t, v_t)$ such that $x_t = \gamma(t)$ and $v_t \in \mathbf{R}^3$. For two variations $\nu = [t \mapsto (x_t, v_t)]$ and $\nu' = [t \mapsto (x_t, v'_t)]$ of γ, define the product

$$\langle \nu, \nu' \rangle = \int_0^1 \langle v_t, v'_t \rangle \, dt$$

Q1. Considers this product to define a formal Riemannian metric on $\mathrm{Paths}(\mathbf{R}^3)$.

Q2. Explicit the energy of a path $[s \mapsto \gamma_s]$ in $\mathrm{Paths}(\mathbf{R}^3)$.

Proof. Let $\mathrm{P} : \mathrm{U} \to \mathrm{Paths}(\mathrm{X})$ be a n-plot. Let us define $g(\mathrm{P})$ a 2-tensor on U by: for all $r \in \mathrm{U}$ and $v, v' \in \mathbf{R}^3$,

$$g(\mathrm{P})_r(v)(v') = \int_0^1 \left\langle \frac{\partial \gamma_r(t)}{\partial r}(v), \frac{\partial \gamma_r(t)}{\partial r}(v') \right\rangle \, dt\,.$$

Let us prove that g is a Riemannian metric on $\mathrm{Paths}(\mathrm{X})$. Consider $g(\mathrm{P} \circ \mathrm{F})$, with F a smooth paramerization in U. Let us denote $\mathrm{F} : s \mapsto r$, $\mathrm{P} : r \mapsto \gamma$ and then $\mathrm{P} \circ \mathrm{F} : s \mapsto r \mapsto \gamma$. We have:

$$\begin{aligned} g(\mathrm{P} \circ \mathrm{F})_s(w)(w') &= \int_0^1 \left\langle \frac{\partial \gamma(t)}{\partial s}(w), \frac{\partial \gamma(t)}{\partial s}(w') \right\rangle \, dt \\ &= \int_0^1 \left\langle \frac{\partial \gamma(t)}{\partial r}\frac{\partial r}{\partial s}(w), \frac{\partial \gamma(t)}{\partial r}\frac{\partial r}{\partial s}(w') \right\rangle \, dt \end{aligned}$$

We remind that for a smooth parametrization $f : x \mapsto y$, where x and y are two real variables, we use indiferently the notations

$$\mathrm{D}(f) \quad \text{or} \quad \mathrm{D}(x \mapsto y) \quad \text{or} \quad \frac{\partial y}{\partial x}.$$

Then for $g \circ f : x \mapsto y \mapsto z$, the chain-rule writes:

$$\mathrm{D}(x \mapsto z) = \mathrm{D}(x \mapsto y \mapsto z) = \mathrm{D}(y \mapsto z) \circ \mathrm{D}(x \mapsto x),$$

or:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} \circ \frac{\partial y}{\partial x}.$$

Therefore, by denoting

$$v = \frac{\partial r}{\partial s}(w) \quad \text{and} \quad v' = \frac{\partial r}{\partial s}(w')$$

we get:

$$\begin{aligned} g(\mathrm{P} \circ \mathrm{F})_s(w)(w') &= \int_0^1 \left\langle \frac{\partial \gamma(t)}{\partial r} \frac{\partial r}{\partial s}(w), \frac{\partial \gamma(t)}{\partial r} \frac{\partial r}{\partial s}(w') \right\rangle dt \\ &= \int_0^1 \left\langle \frac{\partial \gamma(t)}{\partial r}(v), \frac{\partial \gamma(t)}{\partial r}(v') \right\rangle dt \\ = g(\mathrm{P})_{r=\mathrm{Q}(s)} &\left(\frac{\partial \mathrm{Q}(s)}{\partial s}(w) \right) \left(\frac{\partial \mathrm{Q}(s)}{\partial s}(w') \right) = \mathrm{Q}^*(g(\mathrm{P}))_s(w)(w') \end{aligned}$$

Hence, g is a covariant 2-tensor on $\mathrm{Paths}(\mathbf{R}^3)$. It is symmetric because the scalar product is symetric. Now, let $s \mapsto \gamma_s$ be a path in $\mathrm{Paths}(\mathbf{R}^3)$,

$$g(s \mapsto \gamma_s)_s(1)(1) = \int_0^1 \left\langle \frac{\partial \gamma_s(t)}{\partial s}, \frac{\partial \gamma_s(t)}{\partial s} \right\rangle \, dt = \int_0^1 \left\| \frac{\partial \gamma_s(t)}{\partial s} \right\|^2 \, dt$$

Obviously $g(s \mapsto \gamma_s)_s(1)(1)$ is positive. Now

$$g(s \mapsto \gamma_s)_s(1)(1) = 0 \quad \Rightarrow \quad \left\| \frac{\partial \gamma_s(t)}{\partial s} \right\|^2 = 0.$$

then

$$\frac{\partial \gamma_s(t)}{\partial s} = 0 \quad \Rightarrow \quad \gamma_s = \gamma.$$

The path $s \mapsto \gamma_s$ is constant. Thus, for all 1-forms α on $\mathrm{Paths}(\mathbf{R}^3)$, $\alpha(s \mapsto \gamma) = 0$. Differential forms vanish on constant plots (op. cit. Ex. 96). Therefore, the tensor g defined on $\mathrm{Paths}(\mathbf{R}^3)$ is positive and definite, it is a diffeological Riemannian metric according to the definition above. Hence,

$$\mathrm{E}(s \mapsto \gamma_s) = \frac{1}{2} \int_0^1 ds \int_0^1 dt \left\| \frac{\partial \gamma_s(t)}{\partial s} \right\|^2$$

is the energy of the path $s \mapsto \gamma_s$ in $\mathrm{Paths}(\mathbf{R}^3)$. ▶

A Few Half-Lines

In this note we shall talk about a few diffeologies that appeared, equipping the half-line $[0,\infty[$.

As you may know, the half-line $[0,\infty[\subset \mathbf{R}$ can be equipped with the subset diffeology [**TB**, § 1.33], that is, a plot in $[0,\infty[$ is just a smooth parametrization in $\mathbf{R}$ taking its values in $[0,\infty[$. Let us denote this space by Δ. Actually, Δ is a *manifold with boundary* according to (op. cit. § 4.12, 4.16), the boundary being the point $\{0\}$. Now the set $[0,\infty[$ appears in many other places, as the underlying set for the quotients $\Delta_n = \mathbf{R}^n/\mathrm{O}(n)$ (op. cit. § 1.50, Ex. 50). Indeed, the quotient space[1] Δ_n can be realized as the set $[0,\infty[$ equipped with the pushforward of the usual diffeology of $\mathbf{R}^n$ by the norm-square map $\mathrm{sq}_n : x \mapsto \|x\|^2$. Now, for every integer n, thanks to the inclusion

$$\mathrm{J}_n^{n+1} : \mathbf{R}^n \to \mathbf{R}^{n+1}, \quad \text{defined by} \quad \mathrm{J}_n^{n+1}(x) = \begin{pmatrix} x \\ 0 \end{pmatrix},$$

we get a family of smooth injections on the quotient spaces, denoted by j_n^{n+1}.

These definitions give a *direct system* $\{\Delta_n, j_n^m\}_{n,m\in\mathbf{N}}$ indexed by the integers, where the j_n^m, $m = n+k$, are defined by

$$j_n^{n+k} : \Delta_n \to \Delta_{n+k}, \quad \text{and} \quad j_n^{n+k} = j_{n+k-1}^{n+k} \circ j_{n+k-2}^{n+k-1} \circ \cdots \circ j_{n+1}^{n}.$$

This is summarized by the commutative diagram:

[1]Actually $\Delta_1 = \mathbf{R}/\{\pm 1\}$ is an orbifold (op. cit. § 4.17).

$$
\begin{array}{ccc}
\mathbf{R}^n & \xrightarrow{\;\mathrm{J}_n^{n+1}\;} & \mathbf{R}^{n+1} \\
{\scriptstyle \pi_n}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{n+1}} \\
\Delta_n & \xrightarrow[\;j_n^{n+1}\;]{} & \Delta_{n+1}
\end{array}
$$

Since the category {Diffeology} is stable by the operations of sum (op. cit. § 1.39) and quotient (op. cit. § 1.50), it is possible to define the direct limit (or inductive limit or colimit) of a direct system $\{X_i, f_i^j\}_{i,j\in I}$, where I is an up-directed set of indices,[2] the X_i are diffeological spaces, and the $f_i^j : X_i \to X_j$ are smooth maps such that $f_k^j \circ f_i^k = f_i^j$ and $f_i^i = 1_{X_i}$. By definition

$$\varinjlim X_i = \left(\coprod_{i\in I} X_i\right) /\!\sim,$$

where the equivalence relation is defined by

$$(m, x) \sim (n, y) \Leftrightarrow \exists k, k \geq m, k \geq n \text{ and } j_m^k(x) = j_n^k(y).$$

Then, a plot in $\varinjlim X_i$ is any parametrization $P : U \mapsto \varinjlim X_i$ such that there exists everywhere in U, locally, a plot $Q : V \to \coprod_{i\in I} X_i$ satisfying $\mathrm{class}(Q(r)) = P(r)$ for all $r \in V$, where class is the projection from $\coprod_{i\in I} X_i$ onto its quotient $\varinjlim X_i$. Now, thanks to the definition of the sum of diffeological spaces, that means that everywhere in V, there exists an index i and a domain $W \subset V$ such that $\mathrm{val}(Q \upharpoonright W) \subset X_i$. In other words, there exists everywhere in U, an index i, a domain $W \subset U$, and a plot $Q : W \to X_i$, such that $P(r) = \mathrm{class}_i(Q(r))$, where $\mathrm{class}_i = \mathrm{class} \upharpoonright X_i$ and $r \in W$. So, that is the natural diffeological construction of limit we inherit from the standard definition of sums and quotients.[3] Now, applied to our system above, and after identifying each Δ_n with $[0, \infty[$ equipped with the pushforward of the

[2] In French: un ensemble filtrant croissant d'indices.

[3] I didn't include this definition in the book because I didn't use this construction explicitly in it, and it's something that follows naturally from the definition of sums

smooth diffeology of $\mathbf{R}^n$ by the square map $\mathrm{sq} : x \mapsto \|x\|^2$, the maps j_n^m reduce to $\mathbf{1}_{[0,\infty[}$, and the plots in $\Delta_\infty = \varinjlim(\Delta_n)$ are the parametrizations $\mathrm{P} : \mathrm{U} \to [0,\infty[$ such that everywhere in U, there exists an integer N, a domain $\mathrm{V} \subset \mathrm{U}$, and a smooth map $\mathrm{Q} : \mathrm{V} \to \mathbf{R}^\mathrm{N}$ such that $\mathrm{P}(r) = \|\mathrm{Q}(r)\|^2$ for all $r \in \mathrm{V}$. In other words, there exist N smooth real functions Q_i, defined on V, such that

$$\mathrm{P}(r) = \sum_{i=1}^{\mathrm{N}} \mathrm{Q}_i(r)^2.$$

And that's all for the description of the diffeology of Δ_∞. The plots are the non-negative parametrizations of $\mathbf{R}$ which write locally as a finite sum of squares of smooth real functions.

Now: a day of June, a few years ago, during the Conference in honor of Souriau, I was drinking a Coke on the Cours Mirabeau in Aix-en-Provence with Enxin Wu when he asked if Δ_∞ and Δ could coincide as diffeological spaces? In other words, if any non-negative parametrization of $\mathbf{R}$ could be locally written as a sum of squares of smooth real functions? A good question... We asked Google: "non negative function as sums of squares", unexpectedly the first link appeared on the screen was the paper of Bony and al. [**BBCP06**], which states in its very abstract that:

> "*For $n \geq 4$, there are $\mathcal{C}^\infty$ nonnegative functions f of n variables (...) which are not a finite sum of squares of $\mathcal{C}^2$ functions.*"

It was done. In our words: there exist 4-plots in Δ that cannot be locally lifted smoothly in $\coprod_{n \in \mathrm{N}} \Delta_n$, or: there are plots in Δ which are not plots in Δ_∞. Therefore, if clearly the diffeology of the limit Δ_∞ is finer than the diffeology of Δ, the converse is not true, and these two diffeologies on $[0,\infty[$ do not coincide. We have finally the chain of strictly ordered diffeological spaces on the same underlying set $[0,\infty[$,

$$\Delta_1 \prec \Delta_2 \prec \cdots \prec \Delta_\infty \prec \Delta.$$

---

and quotients. But thanks to this note, I've been able to include a paragraph on the definitions of limits, inductive and projective.

1-Forms on Half-Lines

In this note we characterize the differential 1-forms, defined on the various half-lines Δ, Δ_n, Δ_∞, that share the same underlying set $[0,\infty[$.

We consider a series of diffeologies on the set $[0,\infty[$:

(A) Equipped with the subset diffeology, it is a *manifold with boundary* $\{0\}$, [**TB**, § 4.12, 4.16]. It is denote by Δ.

(B) Equipped with the pushforward of the standard diffeology of $\mathbf{R}^n$, $n > 0$, by the norm-square map:

$$\mathrm{Sq} : \mathbf{R}^n \to [0,\infty[\quad \text{with} \quad \mathrm{Sq} : x \mapsto \|x\|^2,$$

it represents the quotients $\Delta_n = \mathbf{R}^n/\mathrm{O}(n)$ (op. cit. § 1.50, Ex. 50). We denote $\Delta_\infty = \lim_{n\to\infty} \Delta_n$, and when we write Δ_n, we allow n to represent also ∞.

Note 1. There are no two different half-lines above that are diffeomorphic. We recall that $\dim_0(\Delta_n) = n$, $\dim_0(\Delta_\infty) = \infty$ and $\dim_0(\Delta) = \infty$, (op. cit. § Ex. 50,51) and "A few half-lines" in these notes.

Note 2. The choice of the function Sq to characterize the quotient $\mathbf{R}^n/\mathrm{O}(n)$ is irrelevant. Every other bijection with the space of orbits could have been used to push forward the standard diffeology of $\mathbf{R}^n$, as explained in (op. cit. § 1.52). For example we could have chosen equivalently $\mathrm{X} \mapsto \|\mathrm{X}\|$, but then the injection $j : [0,\infty[\to \mathbf{R}$ would have not been smooth.

245. The 1-forms are closed. Every differential 1-form defined on Δ_n, Δ_∞, or $\Delta \subset \mathbf{R}$, is closed. Moreover, every half-line is contractible. Therefore, every 1-form is exact.

Note. In the case $n = 1$, the 1-forms are closed simply because the dimension of Δ_1 is 1 (op. cit. § 6.39). But because the dimension of Δ_n and Δ at the origin are n or ∞, the argument of the dimension doesn't apply so simply.

☞ Proof. Any of these half-lines has $[0,\infty[$ as underlying space, only the diffeology change. In each case, the subset $]0,\infty[$ is D-open. In each case, the diffeology induced on $]0,\infty[$ is the standard diffeology. The difference of behavior in the diffeology happens only on the neighborhood of $\{0\}$. Now, let α be a 1-form on a half-line, let P be a m-plot and $\mathrm{U} \subset \mathbf{R}^m$ be its domain. The subset $\mathrm{V} = \mathrm{P}^{-1}(]0,\infty[)$ is open in U and because $]0,\infty[$ is 1-dimensional $d[\alpha(\mathrm{P} \restriction \mathrm{V})] = 0$. The complementary W of V in U is closed. On its interior $\mathring{\mathrm{W}}$ the plot is constant, therefore $\alpha(\mathrm{P} \restriction \mathring{\mathrm{W}}) = 0$, and then $d[\alpha(\mathrm{P} \restriction \mathring{\mathrm{W}})] = 0$. Now, on the boundary $\partial \mathrm{V} = \bar{\mathrm{V}} - \mathrm{V} = \mathrm{W} - \mathring{\mathrm{W}}$, every point r is a limit $\lim_{n\to\infty} r_n$, with $r_n \in \mathrm{V}$. Thus, by continuity, for all $\xi, \xi' \in \mathbf{R}^m$, $d[\alpha(\mathrm{P})]_r(\xi,\xi') = \lim_{n\to\infty} d[\alpha(\mathrm{P} \restriction \mathrm{V})]_{r_n}(\xi,\xi') = 0$. Hence, $d\alpha(\mathrm{P}) = 0$ everywhere, that is, $d\alpha = 0$.

Next, about contractibility. Since the radial retraction $x \mapsto sx$ in $\mathbf{R}^n$ is equivariant under the action of $\mathrm{O}(n)$, the quotients $\Delta_n = \mathbf{R}^n/\mathrm{O}(n)$ are contractible, and also the limit Δ_∞. For Δ we have the retraction $\rho_s : t \mapsto s^2 t$. The map $(s,t) \mapsto s^2 t$, defined on $\mathbf{R} \times \Delta$ takes its values in Δ and is smooth. Thus, Δ is contractible. According to (op. cit. § 6.90) every closed differential form on a contractible diffeological space is exact. Therefore, every 1-form on these half-lines is the differential of a smooth function, this function can be normalized by zero at the origin, and then is unique. ▶

246. The case of Δ. Every differential 1-form on the embedded half-line $\Delta \subset \mathbf{R}$ is the restriction of a differential 1-form defined on $\mathbf{R}$. In other words, the natural induction $j : [0,\infty[\to \mathbf{R}$ induces a surjective pullback $j^* : \Omega^1(\mathbf{R}) \to \Omega^1(\Delta)$. That is, for all $\alpha \in \Omega^1(\Delta)$, there exists $a \in \mathcal{C}^\infty(\mathbf{R},\mathbf{R})$ such that $\alpha_x = a(x)\,dx$, for all $x \geq 0$.

☞ Proof. Thanks to (op. cit. § 4.13) we know that a smooth function from Δ to $\mathbf{R}$ is the restriction of a smooth function from $\mathbf{R}$ to $\mathbf{R}$. Together with the proposition 1, that gives the result. ▶

247. The case of Δ_n. The set $[0,\infty[$ is equipped with the pushforward of the standard diffeology of $\mathbf{R}^n$ by the norm-square map Sq. That identifies Δ_n with $\mathbf{R}^n/\mathrm{O}(n)$ by $\mathrm{class}(\mathrm{X}) \simeq \|\mathrm{X}\|^2$. The injection $j : \Delta_n \simeq [0,\infty[\to \mathbf{R}$ is smooth. The pullback $j^* : \Omega^1(\mathbf{R}) \to \Omega^1(\Delta_n)$ is, here again, surjective.[1]

☞ Proof. The smoothness of the injection comes from the smoothness of the square $\mathrm{Sq} : \mathbf{R}^n \to [0,\infty[$, with $\mathrm{Sq}(\mathrm{X}) = \|\mathrm{X}\|^2$. Now, for the same reason than previously, every 1-form α on Δ_n is exact, that is, there exists a function $f \in \mathcal{C}^\infty(\Delta_n,\mathbf{R})$ such that $\alpha = df$. Pulled back on $\mathbf{R}^n$, we have $\mathrm{Sq}^*(\alpha) = \mathrm{Sq}^*(df) = d(f \circ \mathrm{Sq})$. The function $\mathrm{F} = f \circ \mathrm{Sq}$ is smooth and invariant by $\mathrm{O}(n)$. Conversely every smooth function $\mathrm{F} : \mathbf{R}^n \to \mathbf{R}$ that is $\mathrm{O}(n)$-invariant is the pullback, by Sq, of a smooth function f on Δ_n. So, every 1-form on Δ_n is the pushforward of a differential $d\mathrm{F}$, where F is smooth and $\mathrm{O}(n)$-invariant. Let us restrict F to the subspace of the vectors $(x,0)$, $x \in \mathbf{R}$, and let $\mathrm{F}(x)$ for $\mathrm{F}(x,0)$. We have $\mathrm{F}(x) = f(x^2)$, that is, $\mathrm{F}(+x) = \mathrm{F}(-x)$. Thanks to Whitney theorem [**Whi43**], there exists a smooth function $g \in \mathcal{C}^\infty(\mathbf{R},\mathbf{R})$ such that $\mathrm{F}(x) = g(x^2)$. Thus, $f(x^2) = g(x^2)$, in other words: $f = g \upharpoonright [0,\infty[$, f is the restriction of a smooth function to the interval $[0,\infty[$. Thus $\alpha = df = d(g \circ j) = j^*(dg)$; written differently, $\alpha_t = a(t)\,dt$, for all $t \in [0,\infty[$. On $\mathbf{R}^n$, the pullback of α writes,

$$\mathrm{Sq}^*(\alpha)_\mathrm{X} = 2a\left(\|\mathrm{X}\|^2\right)\mathrm{X}\cdot d\mathrm{X} = 2a\left(\|\mathrm{X}\|^2\right)\sum_{i=1}^{n} \mathrm{X}^i d\mathrm{X}^i,$$

where a is a smooth function on $\mathbf{R}$. ▶

248. The 1-forms vanish at the origin. Every differential 1-form α, defined on any half-line Δ or Δ_n or Δ_∞, vanishes at the origin (op. cit.

[1] Note that the inclusion j is smooth injective but not an induction.

§ 6.40). That is, for every 1-plot γ pointed at the origin, $\gamma(0) = 0$, we have $\alpha(\gamma)_0 = 0$.

In other words, the *cotangent space* reduces to $\{0\}$ at the origin, (op. cit. § 6.48); it is equal to $\mathbf{R}$ everywhere else. An interesting question would be to describe the diffeology of the cotangent space, and to study its parasymplectic structure,[2] that is, the struture defined by the differential of its Liouville form (op. cit. § 6.49).

Proof. According to what comes before, in every case the injection j from the half-line into $\mathbf{R}$ is smooth, and the form α is the pullback, by j, of some smooth 1-form $A \in \Omega^1(\mathbf{R})$. Thus, $\alpha(\gamma)_0 = j^*(A)(\gamma)_0 = A(j \circ \gamma)_0$. But $j \circ \gamma(0) = 0$ and $j \circ \gamma(t) \geq 0$ imply $d\gamma(t)/dt \mid_{t=0} = 0$. Therefore, $A(j \circ \gamma)_0 = A_0(d\gamma(t)/dt \mid_{t=0}) = 0$, and $\alpha(\gamma)_0 = 0$. ▶

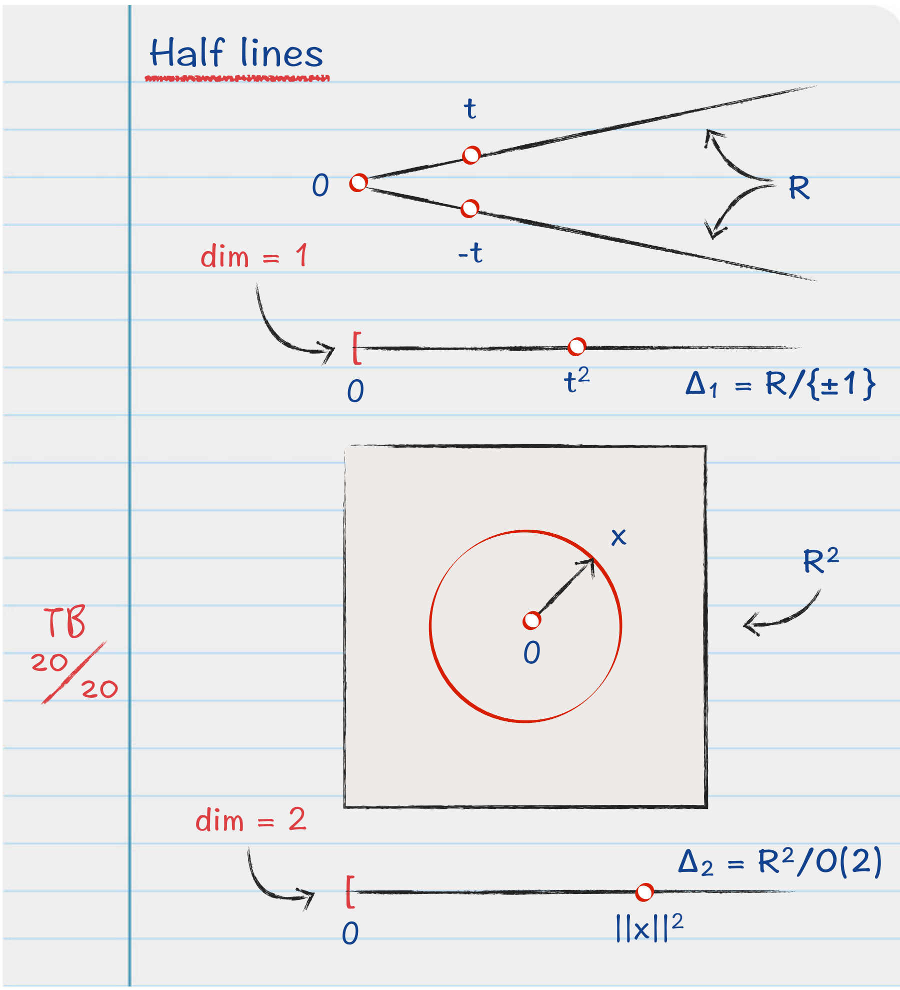
249. The 1-forms as a 1-dimensional module. From what precedes we conclude that, in every case: $\Delta_\star = \Delta$ or Δ_n or Δ_∞, the space of differential 1-forms $\Omega^1(\Delta_\star)$ is a 1-dimensional module on $\Omega^0(\Delta_\star)$, with $dt \restriction [0, \infty[$ as a generator.

250. Gauges on diffeological spaces. There is a notion of volume for diffeological spaces of finite constant dimension, in (op. cit. § 6.44) that almost applies to the half-lines but not completely. First of all, in our case the dimension is not constant (except for the case $n = 1$), but more importantly, the 1-form vanishes at the origin, and volumes are assumed to be nowhere vanishing. Nevertheless, in every case above, the space of 1-forms is a 1-dimensional module on the space of smooth functions, and that is an important remark. That leads to the introduction of the concept of *k-jauge* on a diffeological space, which is slightly different from the concept of volume, but pursues the same idea:

Definition. We call a *k-gauge* on a diffeological space X, any k-form generating $\Omega^k(X)$ as a 1-dimensional module on $\Omega^0(X)$.

[2] Terminology introduced in "Example of Singular Reduction in Symplectic Diffeology".

In our case, for every half-line X, the pullback $j^*(dt)$, where j is the smooth injection of $[0, \infty[$ into $\mathbf{R}$, is a generator of $\Omega^1(X)$. The concept of k-gauge on diffeological spaces worth being studied. There are a few questions around it that need to be answered.



1-Forms on the Subset Half-Line

We revisit the fact that 1-forms on $[0, \infty[\subset \mathbf{R}$ are the restrictions of smooth 1-forms defined on a neighborhood of the half-line in $\mathbf{R}$.

This is a previous result cited in “1-Forms on Half Lines” about the half-line subset $[0, \infty[\subset \mathbf{R}$, that every of its 1-forms are the restriction of some 1-form on $\mathbf{R}$. But in this note, we give a direct proof using Whithey theorem [**Whi43**, Theorem 1 and final remark] (see facsimile in lecture “Local Diffeology, Modeling”).

251. Proposition. Let $\Delta = [0, \infty[\subset \mathbf{R}$ equipped with the subset diffeology. Let $\alpha \in \Omega^1(\Delta)$, then there exists a smooth 1-form $\bar{\alpha}$ defined on some neighborhood of Δ such that $\alpha = \bar{\alpha} \upharpoonright \Delta$. In other words, there exists a smooth function $a \in \mathcal{C}^\infty(]-\varepsilon, \infty[)$, $\varepsilon > 0$, such that α is the restriction of $a(x)dx$ to $[0, \infty[$. In other words, for any n-plot $\mathrm{P} : r \mapsto x_r$ in Δ,

$$\alpha(\mathrm{P})_r(\delta r) = a(\mathrm{P}(r))\frac{\partial x_r}{\partial r}(\delta r),$$

where, $n \in \mathrm{N}$, r belongs to the domain of P and $\delta r \in \mathbf{R}^n$.

Proof. Since $]0, \infty[\subset [0, \infty[$ inherits the usual smooth diffeology, let $\alpha \upharpoonright]0, \infty[= f(x)dx$, and $f \in \mathcal{C}^\infty(]0, \infty[, \mathbf{R})$.

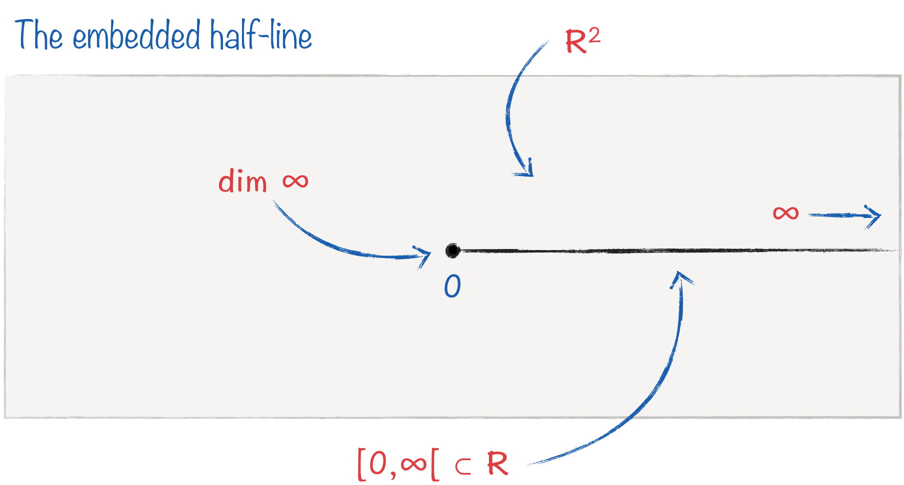
Now, let $\mathrm{sq} : t \mapsto t^2$, then $\mathrm{sq}^*(\alpha)_t(\delta t) = \alpha(t \mapsto t^2)_t(\delta t) = \mathrm{F}(t)\delta t$, with $\mathrm{F} \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$. And for all $t \neq 0$, $\mathrm{F}(t) = f(t^2) \times 2t$.

Next, $\mathrm{sq}^*(\alpha)$ is invariant by $-1 : t \mapsto -t$, thus $(-1)^*(\mathrm{sq}^*(\alpha))_t(\delta t) = \mathrm{sq}^*(\alpha)_t(\delta t)$. That is, $-\mathrm{F}(-t) = \mathrm{F}(t)$ and then $\mathrm{F}(0) = 0$. Hence, there exists a smooth function $\phi \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$ such that $F(t) = 2t\phi(t)$, for all $t \in \mathbf{R}$, and then $f(t^2) = \phi(t)$ for all $t \neq 0$. Therefore, the function ϕ is even, we can apply the Whitney theorem [**Whi43**] :

Theorem. (Whitney) *An even function $f(x) = f(-x)$, defined on a neighborhood of the origin, may be written as $g(x^2)$. If f is smooth, g may be made smooth.*

There exists then a smooth function g defined on a neighborhood of the origin such that $f(t^2) = g(t^2)$. That is, $f = g \restriction\,]0, \infty[$. Let us then define $\bar\alpha = g(x)dx$, $\bar\alpha$ is a smooth 1-form defined on an open neighborhood of $[0, \infty[$, and $\alpha \restriction\,]0, \infty[\, = \bar\alpha \restriction\,]0, \infty[$.

Now, let γ be any path in $[0, \infty[$. Let $\mathcal{O} = \gamma^{-1}(]0, \infty[)$, $\mathcal{O} \subset \mathbf{R}$ is open, and on this open subset $\bar\alpha(\gamma) = \alpha(\gamma)$. Hence, by continuity $\bar\alpha(\gamma) = \alpha(\gamma)$ on the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ (since $\bar\alpha(\gamma)$ and $\alpha(\gamma)$ are smooth). But γ is constant (and equal to 0) on $\mathbf{R} - \overline{\mathcal{O}}$, then $\bar\alpha(\gamma)$ and $\alpha(\gamma)$ both vanishe on $\mathbf{R} - \overline{\mathcal{O}}$. Thus $\alpha(\gamma) = \bar\alpha(\gamma)$ on the whole $\mathbf{R}$. Therefore, since $\bar\alpha$ and α coincide on the 1-plots, they coincide as 1-forms [TB, § 6.37]. ▶



Cotangent Space of the Half-Line

We investigate the nature and structure of the cotangent space of the embedded half-line $\Delta = [0, \infty[\subset \mathbf{R}$.

Consider the half-line $\Delta = [0, \infty[\subset \mathbf{R}$, equipped with the subset diffeology. We have seen above, in "1-Forms on Half-Lines", that every 1-form α on Δ is the pullback of some 1-form on $\mathbf{R}$. In other words, there exists a smooth function $a \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$ such that

$$\alpha(\mathrm{P})_r(\delta r) = a(\mathrm{P}(r)) \frac{\partial \mathrm{P}(r)}{\partial r} \delta r,$$

where P is some plot in Δ, $r \in \mathrm{dom}(\mathrm{P})$ and δr is a vector of $\mathbf{R}^n$, with n the dimension of the plot P. One can also write

$$\alpha(\mathrm{P})_r(\delta r) = a(x)\delta x,$$

with $\mathrm{P} : r \mapsto x$ and $\delta x = \mathrm{D}(\mathrm{P})_r(\delta r)$. We have also seen that the value of α at any point $x \neq 0$ is given by $a(x)$, and that the value of α at 0 is 0. Therefore, the map

$$\pi : \Delta \times \Omega^1(\Delta) \to \mathbf{R} \times \mathbf{R}, \quad \text{defined by} \quad \pi(x, \alpha) = (x, xa(x)),$$

gives a representation of the cotangent space $\mathrm{T}^*(\Delta)$ [**TB**, 6.48], where its image is equipped with the pushforward diffeology. Indeed, if $\pi(x, \alpha) = \pi(x', \alpha')$, then $x = x'$ and if $x \neq 0$, then $a(x) = a(x')$. If $x = 0$, then for whatsoever α and α', $\pi(0, \alpha) = (0, 0) = \pi(0, \alpha')$. Since every 1-form vanishes at the origin, $(0, 0) = \pi(0, \alpha)$ represents the value of α at $x = 0$, for all α.

It's now time to investigate the diffeology on the set

$$\mathrm{val}(\pi) = \{(0,0)\} \bigcup]0,\infty[\times \mathbf{R}$$

when it is equipped with the pushforward diffeology by π.

252. Real functions vanishing at the origin. Let $f \in \mathcal{C}^\infty(\mathbf{R},\mathbf{R})$ such that $f(0) = 0$. Then, there exists $\varphi \in \mathcal{C}^\infty(\mathbf{R}\times\mathbf{R})$ such that $f(x) = x\varphi(x)$, for all $x \in \mathbf{R}$.

Let $\mathcal{C}^\infty_0(\mathbf{R},\mathbf{R})$ be the subset of real smooth functions vanishing at the origin, equipped with the induced diffeology of the functional diffeology on $\mathcal{C}^\infty(\mathbf{R}\times\mathbf{R})$. Then, the map

$$j : \mathcal{C}^\infty_0(\mathbf{R},\mathbf{R}) \to \mathcal{C}^\infty(\mathbf{R},\mathbf{R}) \quad \text{defined by} \quad j(f) = \varphi : x \mapsto \frac{f(x)}{x}$$

is a diffeomorphism.

Proof. First of all, if $x \neq 0$, $\varphi(x) = f(x)/x$. For $x = 0$, φ is extended by continuity and $\varphi(0) = f'(0)$. Next, it is clear that the map $j : f \mapsto \varphi$ is bijective. Now let us consider a plot $r \mapsto f_r$ in $\mathcal{C}^\infty_0(\mathbf{R},\mathbf{R})$. Applying the Taylor's formula with rest [**Die70a**, § 8.14.3], we get, since $f_r(0) = 0$ for all r:

$$f_r(x) = xf_r'(0) + x^2 \int_0^1 (1-t)f_r''(xt)dt.$$

And since $\varphi_r(x) = f_r(x)/x$, we get

$$\varphi_r(x) = f_r'(0) + x \int_0^1 (1-t)f_r''(xt)dt.$$

This expression shows clearly, for f constant in r, that φ is smooth. It shows, moreover, that $(r,x) \mapsto \varphi_r(x)$ is smooth, that is, $r \mapsto \varphi_r$ is a plot of $\mathcal{C}^\infty(\mathbf{R},\mathbf{R})$. Conversely, if $r \mapsto \varphi_r$ is a plot, then $r \mapsto [x \mapsto x\varphi_r(x)]$ is obviously a plot. Therefore, æ is a diffeomorphism. ▶

253. Representing $\mathrm{T}^*(\Delta)$. Let us factorize the projection $\pi : (x,\alpha) \to (x, a(x))$ by

$$(x,\alpha) \mapsto (x, f = [x \mapsto xa(x)]) \mapsto (x, f(x) = (x, xa(x))).$$

Thanks to the first article, the first arrow $(x,a) \mapsto (x, f = [x \mapsto xa(x)])$ is a diffeomorphism from $\Delta\times\Omega^1(\Delta)$ to $\Delta\times\mathcal{C}^\infty_0(\Delta,\mathbf{R})$. The second

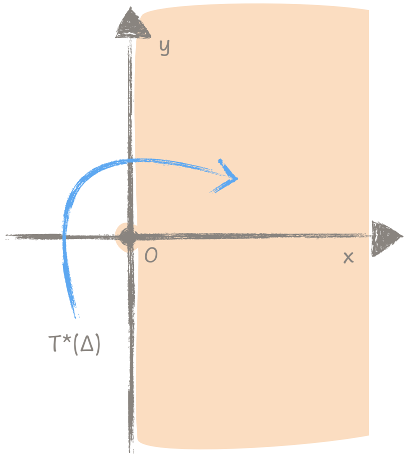


Figure 36. The representation of $\mathrm{T}^*(\Delta)$.

arrow

$$\mathrm{ev} : (x, f) \mapsto (x, f(x)),$$

defined on $\Delta \times \mathcal{C}_0^\infty(\Delta, \mathbf{R})$ is an *evaluation map*.

1) The cotangent space $\mathrm{T}^*(\Delta)$ is diffeomorphic to $\{(0,0)\} \bigcup]0, \infty[\times \mathbf{R}$, equipped with the pushforward diffeology by ev.

2) This diffeology, on $\{(0,0)\} \bigcup]0, \infty[\times \mathbf{R}$, is not the subset diffeology of $\mathbf{R}^2$, it is strictly finer.

Proof. Indeed, consider the 1-plot $\gamma : t \mapsto (x(t) = t^2, y(t) = t)$ of $\{(0,0)\} \bigcup]0, \infty[\times \mathbf{R}$ for the subset diffeology. Assume that there exists a local lift of γ in $\Delta \times \mathcal{C}_0^\infty(\Delta, \mathbf{R})$ near 0, that is, a smooth path $t \mapsto f_t$ in $\mathcal{C}_0^\infty(\mathbf{R}, \mathbf{R})$ such that $y(t) = f_t(x(t))$. According to the article 1, there exists a smooth path $t \mapsto \mathrm{a}_t$ such that $f_t(x) = x\mathrm{a}_t(x)$. Thus, $y(t) = x(t)\mathrm{a}_t(x(t))$, that is, $t = t^2\mathrm{a}_t(t^2)$, i.e. $\mathrm{a}_t(t^2) = 1/t$. And that is not possible since $t \mapsto \mathrm{a}_t(t^2)$ is defined for $t = 0$ (and moreover smooth). Therefore, ev is not a subduction on $\{(0,0)\} \bigcup]0, \infty[\times \mathbf{R}$ for the subset diffeology. Said differently, not all plots of the subset diffeology are plots for the cotangent diffeology. The cotangent diffeology is then finer than the subset diffeology. ▶

1-Forms on Half-Spaces

In this note we shall see that every differential 1-form on the half-space $H^n = [0, \infty[\times \mathbf{R}^{n-1}$ is the restriction of a smooth 1-form on $\mathbf{R}^n$.

This is a (almost) straightforward generalisation of the proposition of "1-Forms On The Half Line" in the previous article. And with a little bit of rewriting, this proof applies to all differential k-forms on half-spaces.

254. Differential 1-form on half-space. Let $H^n = [0, \infty[\times \mathbf{R}^{n-1}$ be the half n-space, equipped with the subset diffeology. Let $\alpha \in \Omega^1(H^n)$ be a differential 1-form on H^n. Then, there exists a smooth 1-form $\bar{\alpha}$ defined on some neighborhood of $H^n \subset \mathbf{R}^n$ such that $\alpha = \bar{\alpha} \restriction H^n$.

☞ Proof. Since $]0, \infty[\times \mathbf{R}^{n-1} \subset [0, \infty[\times \mathbf{R}^{n-1}$ inherits the usual smooth diffeology,

$$\alpha \restriction]0, \infty[\times \mathbf{R}^{n-1} = a(x, y) dx + \sum_{i=1}^{n-1} b_i(x, y) dy_i$$

where $(x, y) \in]0, \infty[\times \mathbf{R}^{n-1}$ and $a, b_i \in \mathcal{C}^\infty(]0, \infty[\times \mathbf{R}^{n-1}, \mathbf{R})$.

Now, let

$$\mathrm{sq}_1 : (t, y) \mapsto (t^2, y),$$

then

$$\mathrm{sq}_1^*(\alpha)_{(t,y)}(\delta t, \delta y)) \quad = \quad \alpha((t, y) \mapsto (t^2, y))_{(t,y)}(\delta t, \delta y)$$

$$= \quad A(t,y)\delta t + \sum_{i=1}^{n-1} B_i(t,y)\delta y_i,$$

with $A, B_i \in \mathcal{C}^\infty(\mathbf{R}^n, \mathbf{R})$. And for all $t \neq 0$,

$$A(t,y) = 2t\, a(t^2, y) \text{ and } B_i(t,y) = b_i(t^2, y).$$

Next, $\mathrm{sq}_1^*(\alpha)$ is invariant by $(-1,1) : (t,y) \mapsto (-t,y)$, thus

$$\begin{aligned} \mathrm{sq}_1^*(\alpha)_{(t,y)}(\delta t, \delta y) &= (-1,1)^*(\mathrm{sq}_1^*(\alpha))_{(t,y)}(\delta t, \delta y) \\ &= \mathrm{sq}_1^*(\alpha)_{(-t,y)}(-\delta t, \delta y) \\ &= A(-t,y)(-\delta t) + \sum_{i=1}^{n-1} B_i(-t,y)\delta y. \end{aligned}$$

Thus, $-A(-t,y) = A(t,y)$ and $B_i(t,y) = B_i(-t,y)$. In particular, $A(0,y) = 0$. Hence, there exists a smooth function $\underline{A} \in \mathcal{C}^\infty(\mathbf{R}^n, \mathbf{R})$ such that $A(t,y) = 2t\underline{A}(t,y)$, for all $t \in \mathbf{R}$. Thus, $a(t^2,y) = \underline{A}(t,y)$. Now, $\underline{A}$ is even in t, as well as the B_i. We can then apply the Hassler Whitney theorem [**Whi43**, Theorem 1 and final remark] (See facsimile in lecture "Local Diffeology, Modeling"), stated as follows:

Theorem. (Whitney) *If a smooth function $f(t,x)$ is even in t, $f(t,x) = f(-t,x)$, then there exists a smooth function $g(t,x)$ such that $f(t,x) = g(t^2,x)$.*

Hence, there exists a smooth function $\underline{a}(t,y)$ such that $\underline{A}(t,y) = \underline{a}(t^2,y)$, and there exists $(n-1)$ smooth functions $\underline{b}_i$ such that $B_i(t,y) = \underline{b}_i(t^2,y)$. We have then, for all $t > 0$, $a(t,y) = \underline{a}(t,y)$ and $b_i(t,y) = \underline{b}_i(t,y)$.

Let us then define $\bar{\alpha}$ on $\mathbf{R}^n$,

$$\bar{\alpha} = \underline{a}(x,y)dx + \sum_{i=1}^{n-1} \underline{b}_i(t,y)dy_i.$$

The form $\bar{\alpha}$ is a smooth 1-form defined on an open neighborhood of H^n, and $\alpha \restriction]0,\infty[\times \mathbf{R}^{n-1} = \bar{\alpha} \restriction]0,\infty[\times \mathbf{R}^{n-1}$. Let us prove now that α and $\bar{\alpha}$ coincide on the whole H^n. Since α and $\bar{\alpha} \restriction H^n$ are two differential 1-forms on H^n, it is enough to show that they take the same value on any smooth path.

Let γ be any path in $[0,\infty[\times \mathbf{R}^{n-1}$. Let $\mathcal{O} = \gamma^{-1}(]0,\infty[\times \mathbf{R}^{n-1})$, $\mathcal{O} \subset \mathbf{R}$ is open, and on this open subset $\bar{\alpha}(\gamma) = \alpha(\gamma)$. Hence, by continuity $\bar{\alpha}(\gamma) = \alpha(\gamma)$ on the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ (since $\bar{\alpha}(\gamma)$ and $\alpha(\gamma)$ are smooth). But on the open subset $\mathbf{R} - \overline{\mathcal{O}}$, γ takes its values in $\partial H^n = \{0\} \times \mathbf{R}^{n-1}$; $\gamma \restriction \mathbf{R} - \overline{\mathcal{O}}$ is a plot of the boundary ∂H^n. Let $i_2 : \mathbf{R}^{n-1} \to \partial H^n$, $i_2(y) = (0, y)$. Then, $i_2^*(\alpha)$ and $i_2^*(\bar{\alpha})$ are both 1-forms on $\mathbf{R}^{n-1}$. Let us prove that they coincide. On the one hand

$$i_2^*(\bar{\alpha})_y(\delta y) = \sum_{i=1}^{n-1} \underline{b}_i(0, y)\delta y.$$

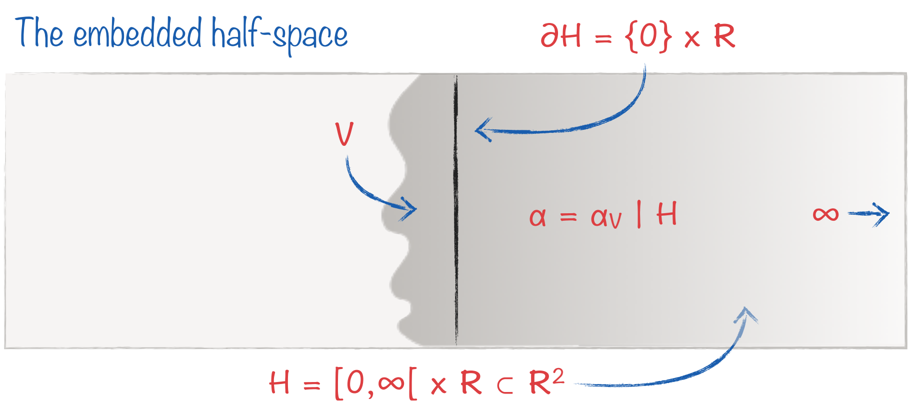
On the other hand, let us notice that

$$i_2 = \mathrm{sq}_1 \circ i_2 : y \mapsto (0, y) \mapsto (0^2, y).$$

Thus, $i_2^*(\alpha) = i_2^*(\mathrm{sq}_1^*(\alpha))$ and then $i_2^*(\alpha)_y(\delta y) = \mathrm{sq}_1^*(\alpha)_{(0,y)}(0, \delta y)$. But,

$$\begin{aligned} \mathrm{sq}_1^*(\alpha)_{(0,y)}(0, \delta y) &= A(0, y) \times 0 + \sum_{i=1}^{n-1} B_i(0, y)\delta y_i \\ &= \sum_{i=1}^{n-1} \underline{b}_i(0, y)\delta y_i, \end{aligned}$$

since $B_i(t, y) = \underline{b}_i(t^2, y)$. Hence α and $\bar{\alpha}$ coincide on ∂H^n and then $\bar{\alpha}(\gamma)$ and $\alpha(\gamma)$ coincide everywhere. Therefore, since $\bar{\alpha}$ and α coincide on the 1-plots in H^n, they coincide as 1-forms [TB, § 6.37], and then, $\alpha = \bar{\alpha} \restriction H^n$. ▶



p-Forms on Half-Spaces

We shall see that every differential p-form on the half-space $\mathrm{H}^n = [0,\infty[\times\mathbf{R}^{n-1}$ is the restriction of a smooth p-form on $\mathbf{R}^n$.

This note is a natural development of the previous "1-Forms on half-spaces".

255. Differential p-forms on half-spaces. Let $\mathrm{H}^n = [0,\infty[\times\mathbf{R}^{n-1}$ be the half n-space, equipped with the subset diffeology. Let $\omega \in \Omega^p(\mathrm{H}^n)$ be a differential p-form on H^n. Then, there exists a smooth p-form $\bar\omega$ defined on some neighborhood of $\mathrm{H}^n \subset \mathbf{R}^n$ such that $\omega = \bar\omega \restriction \mathrm{H}^n$.

Proof. Since $\mathring{\mathrm{H}}^n =]0,\infty[\times\mathbf{R}^{n-1} \subset \mathrm{H}^n = [0,\infty[\times\mathbf{R}^{n-1}$ inherits the usual smooth diffeology,

$$\begin{aligned}\omega \restriction \mathring{\mathrm{H}}^n &= \sum_{1<j<\cdots<k} a_{1j\ldots k}(x,y)dx \wedge dy_j \wedge \cdots \wedge dy_k \\ &+ \sum_{i<j<\cdots<k} b_{ij\ldots k}(x,y)dy_i \wedge dy_j \wedge \cdots \wedge dy_k,\end{aligned}$$

where $(x,y) \in \mathring{\mathrm{H}}^n$ and $a_{1j\ldots k}, b_{ij\ldots k} \in \mathcal{C}^\infty(\mathring{\mathrm{H}}^n, \mathbf{R})$.

Now, let

$$\mathrm{sq}_1 : (t,y) \mapsto (t^2, y),$$

then let

$$\begin{aligned}\mathrm{sq}_1^*(\omega) &= \omega((t,y) \mapsto (t^2,y)) \\ &= \sum_{1<j<\cdots<k} A_{1j\ldots k}(t,y)dt \wedge dy_j \wedge \cdots \wedge dy_k\end{aligned}$$

$$+ \sum_{i<j<\cdots<k} \mathrm{B}_{ij\ldots k}(t,y) dy_i \wedge dy_j \wedge \ldots dy_k,$$

with $\mathrm{A}_{1j\ldots k}, \mathrm{B}_{ij\ldots k} \in \mathcal{C}^\infty(\mathbf{R}^n, \mathbf{R})$. And for all $t \neq 0$,

$$\mathrm{A}_{1j\ldots k}(t,y) = 2t\, \mathrm{a}_{1j\ldots k}(t^2, y) \text{ and } \mathrm{B}_{ij\ldots k}(t,y) = b_{ij\ldots k}(t^2, y).$$

Next, $\mathrm{sq}_1^*(\omega)$ is invariant by $\varepsilon : (t,y) \mapsto (-t,y)$, thus

$$\begin{aligned} \mathrm{sq}_1^*(\omega) &= \varepsilon^*(\mathrm{sq}_1^*(\omega)) \\ &= \sum_{1<j<\cdots<k} \mathrm{A}_{1j\ldots k}(-t,y)(-dt \wedge dy_i \wedge \cdots \wedge dy_k) \\ &+ \sum_{i<j<\cdots<k} \mathrm{B}_{ij\ldots k}(-t,y) dy_i \wedge dy_j \wedge \cdots \wedge dy_k. \end{aligned}$$

Thus, $-\mathrm{A}_{1j\ldots k(-t,y)} = \mathrm{A}_{1j\ldots k}(t,y)$ and $\mathrm{B}_{ij\ldots k}(t,y) = \mathrm{B}_{ij\ldots k}(-t,y)$. In particular, $\mathrm{A}_{1j\ldots k}(0,y) = 0$. Hence, there exists a smooth function $\underline{\mathrm{A}}_{1j\ldots k} \in \mathcal{C}^\infty(\mathbf{R}^n, \mathbf{R})$ such that $\mathrm{A}_{1j\ldots k}(t,y) = 2t\underline{\mathrm{A}}_{1j\ldots k}(t,y)$, for all $t \in \mathbf{R}$. Thus, for all $t \neq 0$, $\mathrm{a}_{1j\ldots k}(t^2,y) = \underline{\mathrm{A}}_{1j\ldots k}(t,y)$. Now, $\underline{\mathrm{A}}_{1j\ldots k}$ is even in t, as well as the $\mathrm{B}_{ij\ldots k}$. We can then apply the Hassler Whitney theorem [**Whi43**, Theorem 1 and final remark], stated as follows:

<u>Theorem. (Whitney)</u> *If a smooth function $f(t,x)$ is even in t, $f(t,x) = f(-t,x)$, then there exists a smooth function $g(t,x)$ such that $f(t,x) = g(t^2,x)$.*

Hence, there exist smooth functions $\underline{\mathrm{a}}_{1j\ldots k}(t,y)$ and $\underline{b}_{ij\ldots k}(t,y)$, such that $\underline{\mathrm{A}}_{1j\ldots k}(t,y) = \underline{\mathrm{a}}_{1j\ldots k}(t^2,y)$ and $\mathrm{B}_{ij\ldots k}(t,y) = \underline{b}_{ij\ldots k}(t^2,y)$. Then, for all $t > 0$, $\mathrm{a}_{1j\ldots k}(t,y) = \underline{\mathrm{a}}_{1j\ldots k}(t,y)$ and $b_{ij\ldots k}(t,y) = \underline{b}_{1j\ldots k}(t,y)$.

Let us then define $\bar\omega$ on $\mathbf{R}^n$,

$$\begin{aligned} \bar\omega &= \sum_{1<j<\cdots<k} \underline{\mathrm{a}}_{1j\ldots k}(x,y) dx \wedge dy_j \wedge \cdots \wedge dy_k \\ &+ \sum_{i<j<\cdots<k} \underline{b}_{ij\ldots k}(t,y) dy_i \wedge dy_j \wedge \cdots \wedge dy_k. \end{aligned}$$

The form $\bar\omega$ is a smooth p-form defined on an open neighborhood of H^n, and $\omega \restriction \mathring{\mathrm{H}}^n = \bar\omega \restriction \mathring{\mathrm{H}}^n$. Let us prove now that ω and $\bar\omega$ coincide on the whole H^n. Since ω and $\bar\omega \restriction \mathrm{H}^n$ are two differential p-forms on

H^n, it is sufficient to check that they take the same value on every smooth p-path.

Let σ be any p-path in H^n. Let $\mathcal{O} = \sigma^{-1}(\mathring{H}^n)$, $\mathcal{O} \subset \mathbf{R}^p$ is open, and on this open subset $\bar{\omega}(\sigma) = \omega(\sigma)$. Hence, by continuity $\bar{\omega}(\sigma) = \omega(\sigma)$ on the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ (since $\bar{\omega}(\sigma)$ and $\omega(\sigma)$ are smooth). But on the open subset $\mathbf{R}^p - \overline{\mathcal{O}}$, σ takes its values in $\partial H^n = \{0\} \times \mathbf{R}^{n-1}$; $\bar{\sigma} = \sigma \restriction \mathbf{R}^p - \overline{\mathcal{O}}$ is a plot of the boundary ∂H^n. Let $i : \mathbf{R}^{n-1} \to \partial H^n$, $i(y) = (0, y)$. Then, $i^*(\omega)$ and $i^*(\bar{\omega})$ are both p-forms on $\mathbf{R}^{n-1}$. Let us prove that they coincide. On the one hand

$$i^*(\bar{\omega}) = \sum_{i<j<\cdots<k} \underline{b}_{ij\ldots k}(0, y) dy_i \wedge dy_j \wedge \cdots \wedge dy_k.$$

On the other hand, let us notice that

$$i = \mathrm{sq}_1 \circ i : y \mapsto (0, y) \mapsto (0^2, y).$$

Thus, $i^*(\omega) = i^*(\mathrm{sq}_1^*(\omega))$ and then

$$i^*(\omega)_y(\delta_1 y, \ldots, \delta_{p-1} y) = \mathrm{sq}_1^*(\omega)_{(0,y)}(0, \delta_1 y, \ldots, \delta_{p-1} y).$$

Hence,

$$\begin{aligned} \mathrm{sq}_1^*(\omega) &= \sum_{i<j<\cdots<k} B_{ij\ldots k}(0, y) dy_i \wedge dy_j \wedge \cdots \wedge dy_k \\ &= \sum_{i=1}^{n-1} \underline{b}_{ij\ldots k}(0, y) dy_i \wedge dy_j \wedge \cdots \wedge dy_k, \end{aligned}$$

since $A(0, y) = 0$ and $B_{ij\ldots k}(t, y) = \underline{b}_{ij\ldots k}(t^2, y)$. Hence ω and $\bar{\omega}$ coincide on ∂H^n and then $\bar{\omega}(\sigma)$ and $\omega(\sigma)$ coincide everywhere. Therefore, since $\bar{\omega}$ and ω coincide on the p-plots, they coincide as p-forms [TB, § 6.37], and then, $\omega = \bar{\omega} \restriction H^n$. ▶

p-Forms on Corners

In this note we shall see that, for the subset diffeology, differential forms defined on half-spaces or corners of Euclidean spaces, are the restrictions of a differential forms defined on an open neighborhood of the corner in the ambient Euclidean space.

Heuristically, smooth maps from corners $K^n = \{(x_1, \ldots, x_n) \mid x_i \geq 0\}$ into the real line $\mathbf{R}$ are just defined as restrictions of smooth maps, defined on some open neighborhood of the corner [**Cer61**] [**Dou62**] etc. This heuristic becomes a theorem in diffeology where K^n is equipped with the subset diffeology. Indeed, every map from K^n to $\mathbf{R}$ which is smooth composed with any smooth parametrization $P: U \to \mathbf{R}^n$ taking its values in K^n, is the restriction of a smooth maps defined on some open neighborhood of the corner [**TB**, § 4.16].

It is always a progress when a convention, based on mathematicians' intuition, becomes a theorem in a well defined axiomatic. Here the axiomatic is the theory of Diffeology. Noticing that $\mathcal{C}^\infty(K^n, \mathbf{R})$ is just the space of differential 0-forms $\Omega^0(K^n)$, it is legitimate to ask about the behavior of differential k-forms on K^n, that is, $\Omega^k(K^n)$ as it is defined in (op. cit § 6.28). In this note we prove the following theorem stated in (art. 259):

Theorem. *Every differential form on the corner K^n is the restriction of a smooth form on an open neighborhood of K^n in $\mathbf{R}^n$. Precisely,* the pullback : $j^* : \Omega^k(\mathbf{R}^n) \to \Omega^k(\mathrm{K}^n)$ is surjective, where j denotes the inclusion from K^n into $\mathbf{R}^n$.

Do we need to remind that a differential k-form on a diffeological space X is a mapping α that associates with each plot P in X, a smooth k-form $\alpha(\mathrm{P})$ on dom(P), such that the smooth compatibility condition $\alpha(\mathrm{F} \circ \mathrm{P}) = \mathrm{F}^*(\alpha(\mathrm{P}))$ is satisfied, where F is any smooth parametrization in dom(P).

64. Smooth structure on corners

256. Corners as diffeologies. We denote by K^n the *corner*

$$\mathrm{K}^n = \{(x_i)_{i=1}^n \in \mathbf{R}^n \mid x_i \geq 0,\ i = 1, \ldots, n\}.$$

And we equip it with the subset diffeology. A plot in K^n is just a regular smooth parametrization in $\mathbf{R}^n$ but taking its values in K^n.

(A) The corner K^n is the diffeological n-power of the half-line $\mathrm{K} = [0, \infty[\subset \mathbf{R}$, equipped with the subset diffeology.

(B) The corner K^n is *embedded* in $\mathbf{R}^n$, and closed. That is, the D-topology of the induction $\mathrm{K}^n \subset \mathbf{R}^n$ coincides with the induced topology[1] of $\mathbf{R}^n$, see (op. cit § 2.13).

(C) Let $\mathrm{X}_0 = \{0\} \subset \mathrm{X}_1 \subset \cdots \subset \mathrm{X}_n = \mathrm{K}^n$ be the natural filtration of K^n, where the *levels* X_j are defined by

$$\mathrm{X}_j = \{(x_i)_{i=1}^n \in \mathrm{K}^n \mid \text{there exists } i_1 < \cdots < i_{n-j} \text{ such that } x_{i_\ell} = 0\}.$$

Then, the *stratum*

$$\mathrm{S}_j = \mathrm{X}_j - \mathrm{X}_{j-1}$$

[1]The standard topology of $\mathbf{R}^n$ is the D-topology of its standard smooth structure.

is the subset of points in $\mathbf{R}^n$ that have j, and only j, coordinates strictly positive. The strata S_j are equipped with the subset diffeology.[2]

$$S_j = \left\{ (x_i)_{i=1}^n \in \mathbf{R}^n \;\middle|\; \begin{array}{l} \text{There exist } i_1 < \cdots < i_j \text{ such that } x_{i_\ell} > 0, \\ \text{and } x_m = 0 \text{ for all } m \notin \{i_1, \ldots, i_j.\} \end{array} \right\}.$$

Then, S_j is D-open in X_j, $j \geq 1$. As a subset of X_j, S_j is the (diffeological) sum of $\binom{n}{j}$ connected components indexed by a string of j ones and $n-j$ zeros.

Proof. For the first item, it's immediately by definition. Considering the second item: for any subset $U \subset K^n$ open for the induced topology, there exists (by definition) an open subset $\mathcal{O} \in \mathbf{R}^n$ such that $U = \mathcal{O} \cap K^n$. Then, for all plots P in K^n, $P^{-1}(U) = P^{-1}(\mathcal{O})$ is open, because plots are continuous. On the other hand, let $U \subset K^n$ be D-open. Then, $\mathrm{sq}^{-1}(U) \subset \mathbf{R}^n$ is open, where $\mathrm{sq}\colon \mathbf{R}^n \to K^n$ is the map $\mathrm{sq}(x_i)_{i=1}^n = (x_i^2)_{i=1}^n$. And $\mathrm{sq}^{-1}(U) \restriction K^n$ is open for the induced topology of $\mathbf{R}^n$. Now, the map sq restricted to K^n is an homeomorphism. Hence, since $U = \mathrm{sq}(\mathrm{sq}^{-1}(U) \restriction K^n)$, U is open for the induced topology of $\mathbf{R}^n$. Therefore the D-topology of the induction coincides with the induced topology, as we claimed.

For the third item: let $x \in X_j$, then the number ν of vanishing coordinates of x is at least $n-j$, *i.e.* $\nu \geq n-1$. Next, if $x \in X_j$ and $x \notin X_{j-1}$, then $\nu \geq n-j$ and $\nu < n-j+1$, thus, $\nu = n-j$. Therefore, $X_j - X_{j-1}$ is the subset of points in $\mathbf{R}^n$ that have exactly $n-j$ coordinates equal to 0 and the other j strictly positive:

Consider now a point $x = (x_1, \ldots, x_n) \in S_j - S_{j-1}$. Since the j non-zero coordinates of x are strictly positive, there exists $\varepsilon > 0$ such that $x_i - \varepsilon > 0$, for all non-zero coordinate of x. The open n-parallelepiped $C_x =]x_1 - \varepsilon, x_1 + \varepsilon[\times \cdots \times]x_n - \varepsilon, x_n + \varepsilon[\subset \mathbf{R}^n$ contains x, and $C_x \cap S_j \subset S_j - S_{j-1}$. Thus,

$$S_j - S_{j-1} = \bigcup_{x \in S_j - S_{j-1}} C_x \cap S_j.$$

[2] Recall that, by transitivity of subset diffeology, to be a subspace of S_ℓ or K^n or of $\mathbf{R}^n$ is identical.

Now, let $P : U \to S_j$ be a plot for the subset diffeology. We have, $P^{-1}(S_j - S_{j-1}) = \cup_{x \in S_j - S_{j-1}} P^{-1}(C_x \cap S_j)$, but $P^{-1}(C_x \cap S_j) = P^{-1}(C_x)$ since $\mathrm{val}(P) \subset S_j$. Next, since P is smooth as a map into $\mathbf{R}^n$ and C_x is open, $P^{-1}(C_x)$ is open and then $P^{-1}(S_j - S_{j-1})$ is open. Therefore, $S_j - S_{j-1}$ is D-open in S_j. ▶

257. Smooth maps on corners. We know that a map $f : K^n \to \mathbf{R}$, is smooth in the sense of diffeology, if and only if it is the restriction of a smooth map F defined on some open neighborhood $\mathcal{O}$ of K^n into $\mathbf{R}$, see (op. cit § 4.16). That is, $f \in \mathcal{C}^\infty(K^n, \mathbf{R})$ if and only if, $f = F \upharpoonright K^n$ and $F \in \mathcal{C}^\infty(\mathcal{O}, \mathbf{R})$.

258. The square function lemma. Let $\mathrm{sq} : \mathbf{R}^n \to K^n$ be the smooth parametrization:

$$\mathrm{sq}(x_1, \ldots, x_n) = (x_1^2, \ldots, x_n^2).$$

Then $\mathrm{sq}^* : \Omega^k(K^n) \to \Omega^k(\mathbf{R}^n)$ is injective. That is, for all $\alpha \in \Omega^k(K^n)$, if $\mathrm{sq}^*(\alpha) = 0$, then $\alpha = 0$.

☞ Proof. Note that each component of $S_j - S_{j-1}$ is isomorphic to $\mathbf{R}^j$. Hence, if $\mathrm{sq}^*(\alpha) = 0$, since $\mathrm{sq} \upharpoonright \mathrm{sq}^{-1}(S_j - S_{j-1})$ is a 2-folds covering over $S_j - S_{j-1}$, $\alpha \upharpoonright S_j - S_{j-1} = 0$. that is, for all plots Q in $S_j - S_{j-1}$, $\alpha(Q) = 0$. Let then, for some $j \geq 1$, $P_j : U_j \to S_j$ be a plot. In view of what precedes, the subset $\mathcal{O}_j = P_j^{-1}(S_j - S_{j-1})$ is open, and $\alpha(P_j \upharpoonright \mathcal{O}_j) = \alpha(P_j) \upharpoonright \mathcal{O}_j = 0$. By continuity, $\alpha(P_j) \upharpoonright \overline{\mathcal{O}}_j = 0$, where $\overline{\mathcal{O}}_j$ is the closure of $\mathcal{O}_j$. Let then $U_{j-1} = U_j - \overline{\mathcal{O}}_j$ and $P_{j-1} = P_j \upharpoonright U_{j-1}$. Then, U_{j-1} is open and $P_{j-1} : U_{j-1} \to S_{j-1}$ is a plot. This construction gives a descending recursion, starting with any plot $P : U \to K^n$, by initializing $P_n = P$, $U_n = U$ and $S_n = K^n$. One has $P_j = P \upharpoonright U_j$, $U_{j-1} \subset U_j$, the recursion ends with a plot P_0 with values in $S_0 = \{0\}$, and $\alpha(P_0) = 0$ since P_0 is constant. Therefore $\alpha = 0$. ▶

259. Differential forms on corners. The previous article (art. 257) deals with smooth real functions on corners, that is, $\Omega^0(K^n)$. It is a particular case of the more general theorem:

Theorem. *Any differential k-form on the corner K^n, equipped with the subset diffeology of $\mathbf{R}^n$, is the restriction of a smooth differential*

k-form defined on some open neighborhood of the corner. Precisely, the pullback : $j^* \colon \Omega^k(\mathbf{R}^n) \to \Omega^k(\mathrm{K}^n)$ is surjective, where j denotes the inclusion from K^n to $\mathbf{R}^n$.

Proof. Let $\omega \in \Omega^k(\mathrm{K}^n)$ and $\mathring{\mathrm{K}}^n = \{(x_i)_{i=1}^n \mid x_i > 0, i = 1, \ldots, n\}$. One has

$$\omega \upharpoonright \mathring{\mathrm{K}}^n = \sum_{i_1 < \cdots < i_k} \mathrm{a}_{i_1 \ldots i_k}(x_1, \ldots, x_n)\ dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

with $i_j = 1, \ldots, n$ and $\mathrm{a}_{i_1 \ldots i_k} \in \mathcal{C}^\infty(\mathring{\mathrm{K}}^n, \mathbf{R})$. Recall that $\mathrm{sq} \colon (x_i)_{i=1}^n \mapsto (x_i^2)_{i=1}^n$, then

$$\mathrm{sq}^*(\omega) = \sum_{i_1 < \cdots < i_k} \mathrm{A}_{i_1 \ldots i_k}(x_1, \ldots, x_n)\ dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where $\mathrm{A}_{i_1 \ldots i_k} \in \mathcal{C}^\infty(\mathbf{R}^n, \mathbf{R})$. Let $\varepsilon_j : (\ldots, x_j, \ldots) \mapsto (\ldots, -x_j, \ldots)$, then $\mathrm{sq} \circ \varepsilon_j = \mathrm{sq}$ and $(\mathrm{sq} \circ \varepsilon_j)^*(\omega) = \varepsilon_j^*(\mathrm{sq}^*(\omega))$, that is, $\mathrm{sq}^*(\omega) = \varepsilon_j^*(\mathrm{sq}^*(\omega))$. Hence,

$$\begin{aligned}\varepsilon_j^*(\mathrm{sq}^*(\omega)) = {} & \sum_{\substack{i_1 < \cdots < i_k \\ i_\ell \neq j}} \mathrm{A}_{i_1 \ldots i_k}(x_1, \ldots, -x_j, \ldots, x_n)\ dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ & - \sum_{i_1 < \cdots \leq j \leq \cdots < i_k} \mathrm{A}_{i_1 \ldots j \ldots i_k}(x_1, \ldots, -x_j, \ldots, x_n)\ dx_{i_1} \wedge \ldots dx_j \cdots \wedge dx_{i_k}.\end{aligned}$$

Then,

$$\begin{aligned}\mathrm{A}_{\substack{i_1 \ldots i_k \\ i_\ell \neq j}}(x_1, \ldots, -x_j, \ldots, x_n) &= \mathrm{A}_{i_1 \ldots i_k}(x_1, \ldots, x_j, \ldots, x_n), \\ \mathrm{A}_{i_1 \ldots j \ldots i_k}(x_1, \ldots, -x_j, \ldots, x_n) &= -\mathrm{A}_{i_1 \ldots j \ldots i_k}(x_1, \ldots, x_j, \ldots, x_n).\end{aligned}$$

Hence,

$$\mathrm{A}_{i_1 \ldots j \ldots i_k}(x_1, \ldots, x_j = 0, \ldots, x_n) = 0.$$

Thus,

$$\mathrm{A}_{i_1 \ldots j \ldots i_k}(x_1, \ldots, x_j, \ldots, x_n) = 2x_j \underline{\mathrm{A}}_{i_1 \ldots j \ldots i_k}(x_1, \ldots, x_j, \ldots, x_n),$$

with $\underline{\mathrm{A}}_{i_1 \ldots j \ldots i_k} \in \mathcal{C}^\infty(\mathbf{R}^n, \mathbf{R})$. Therefore, there are real smooth functions $\hat{\mathrm{A}}_{i_1 \ldots i_k}$ defined on $\mathbf{R}^n$ such that

$$\mathrm{A}_{i_1 \ldots i_k}(x_1, \ldots, x_n) = 2^k x_{i_1} \ldots x_{i_k} \hat{\mathrm{A}}_{i_1 \ldots i_k}(x_1, \ldots, x_n).$$

Now,

$$\mathrm{sq}^*(\omega \upharpoonright \mathring{\mathrm{K}}^n) = \mathrm{sq}^*(\omega) \upharpoonright \{x_i \neq 0\}$$

implies

$$\begin{aligned} &\sum_{i_1<\cdots<i_k} 2^k x_{i_1}\dots x_{i_k} a_{i_1\dots i_k}(x_1^2,\dots,x_n^2)\ dx_{i_1}\wedge\cdots\wedge dx_{i_k} \\ = &\sum_{i_1<\cdots<i_k} 2^k x_{i_1}\dots x_{i_k} \hat{A}_{i_1\dots i_k}(x_1,\dots,x_n)\ dx_{i_1}\wedge\cdots\wedge dx_{i_k}. \end{aligned}$$

Hence,

$$\hat{A}_{i_1\dots i_k}(x_1,\dots,x_n) = a_{i_1\dots i_k}(x_1^2,\dots,x_n^2) \quad \text{for } x_i \neq 0, i = 1,\dots,n.$$

Thus $(x_1,\dots,x_n) \mapsto \hat{A}_{i_1\dots i_k}(x_1,\dots,x_n)$, which belongs to $\mathcal{C}^\infty(\mathbf{R}^n,\mathbf{R})$, is even in each variable. Therefore, according to Schwartz Theorem **[Sch75]**,[3] there exists

$$\underline{a}_{i_1\dots i_k} \in \mathcal{C}^\infty(\mathbf{R}^n,\mathbf{R}),$$

such that

$$\hat{A}_{i_1\dots i_k}(x_1,\dots,x_n) = \underline{a}_{i_1\dots i_k}(x_1^2,\dots,x_n^2).$$

One deduces:

$$\underline{a}_{i_1\dots i_k}(x_1,\dots,x_n) = a_{i_1\dots i_k}(x_1,\dots,x_n), \text{ for all } (x_1,\dots,x_n) \in \mathring{K}^n.$$

Then, defining the k-form $\underline{\omega}$ on $\mathbf{R}^n$ by

$$\underline{\omega} = \sum_{i_1<\cdots<i_k} \underline{a}_{i_1\dots i_k}(x_1,\dots,x_n)\ dx_{i_1}\wedge\cdots\wedge dx_{i_k},$$

one has already

$$\underline{\omega} \restriction \mathring{K}^n = \omega \restriction \mathring{K}^n.$$

Let us show that $\underline{\omega} \restriction K^n = \omega$. That is, let us check that for all plots $P\colon U \to \mathbf{R}^n$, $P^*(\underline{\omega}) = \omega(P)$. Actually, one has

$$\mathrm{sq}^*(\omega) = \mathrm{sq}^*(\underline{\omega} \restriction K^n).$$

Indeed:

$$\begin{aligned} \mathrm{sq}^*(\omega) &= \sum_{i_1\dots i_k} A_{i_1\dots i_k}(x_1,\dots,x_n)\ dx_{i_1}\wedge\cdots\wedge dx_{i_k} \\ &= \sum_{i_1\dots i_k} 2^k x_{i_1}\dots x_{i_k} \hat{A}_{i_1\dots i_k}(x_1,\dots,x_n)\ dx_{i_1}\wedge\cdots\wedge dx_{i_k} \\ &= \sum_{i_1\dots i_k} 2^k x_{i_1}\dots x_{i_k} \underline{a}_{i_1\dots i_k}(x_1^2,\dots,x_n^2)\ dx_{i_1}\wedge\cdots\wedge dx_{i_k}. \end{aligned}$$

[3] Which is a generalisation of a famous Whitney theorem **[Whi43]**

And, on the other hand:

$$\mathrm{sq}^*(\underline{\omega} \restriction \mathrm{K}^n) = \sum_{i_1 \dots i_k} 2^k x_{i_1} \dots x_{i_k} \underline{a}_{i_1 \dots i_k}(x_1^2, \dots, x_n^2)\ dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Thus, $\mathrm{sq}^*(\omega - \underline{\omega} \restriction \mathrm{K}^n) = 0$. Therefore, according to the previous lemma (art. 258), $\omega - \underline{\omega} \restriction \mathrm{K}^n = 0$. And indeed, ω is the restriction of the smooth k-form $\underline{\omega}$ on K^n. ▶

65. Application

260. An exemple of application. Among the possible applications of the theorems above (art. 258) and (art. 259), there is already one worthy of mention. It is about the description of closed 2-form, invariant with respect to the action of a Lie group. As it has been showed in particular in the classification of SO(3)-symplectic manifolds [**Igl84, Igl91**], any closed 2-form form ω on a manifold M, invariant by a compact group[4] G, is characterized by its *moment map* $\mu \colon \mathrm{M} \to \mathcal{G}^*$ (we assume the action Hamiltonian), and for each moment map, a closed 2-form $\varepsilon \in \mathrm{Z}^2(\mathrm{M}/\mathrm{G})$. Let us be precise: the space of closed 2-forms $\mathrm{Z}^2(\mathrm{M})$ is a vector space, the space of G-equivariant maps from M to $\mathcal{G}^*$ is also a vector space. Then, the map associating its moment map[5] μ with each invariant closed 2-form ω is linear. What we claim is that the kernel of this map is exactly $\mathrm{Z}^2(\mathrm{M}/\mathrm{G})$, where M/G is equipped with the quotient diffeology.

Now, if an equivariant map is easy to conceive, it is more problematic for a differential form on the space of orbits, which is generally not a manifold. This is where the above theorem can help, because it happens that M/G is not far to be a manifold with boundary or corners, as show the following example.

Let us consider the simple case of $\mathrm{M} = \mathbf{R}^{2n}$ equipped with the standard symplectic form $\omega = \sum_{i=1}^n dq_i \wedge dp_i$. It is invariant by the group $\mathrm{SO}(2, \mathbf{R})^n$ acting naturally, each factor on its respective

[4]There could be a diffeological generalisation possible to non compact group.

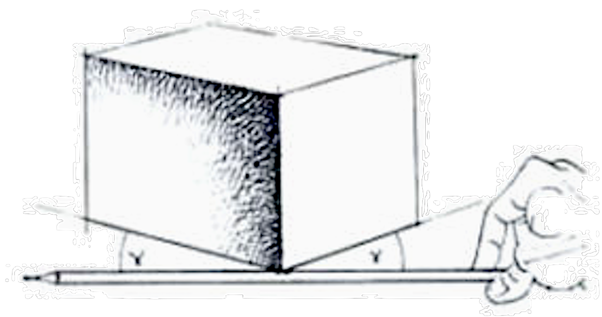
[5]The manifold M is supposed to be connected. To have a unicity of the moment maps we decide to fix their value to 0 at some base point $m_0 \in \mathrm{M}$, for example.

copie of $\mathbf{R}^2$. The (diffeological) quotient space $\mathcal{Q}^n = \mathbf{R}^{2n}/\mathrm{SO}(2,\mathbf{R})^n$ is the n-th power of $\mathcal{Q} = \mathbf{R}^2/\mathrm{SO}(2,\mathbf{R})$. Let $X = (X_i)_{i=1}^n$ with $X_i = (q_i, p_i)$. There is a natural smooth bijection $j_{2n}\colon \mathcal{Q}^n \to K^n$, given by $j_{2n}\colon \mathrm{class}(X) \mapsto (\|X_i\|^2)_{i=1}^n$. It turns out that this smooth bijection[6] induces, by pullback, an injection j_{2n}^* from $\Omega^k(K^n)$ into $\Omega^k(\mathcal{Q}_n)$. Thus, thanks to (art. 259), for each 2-form ε on the quotient $\mathcal{Q}_n$ there exists a 2-form $\underline{\varepsilon}$ on $\mathbf{R}^n$, such that $\varepsilon = j_{2n}^*(\underline{\varepsilon})$. And the 2-form ω is characterized by μ and $\underline{\varepsilon} \restriction K^n$, with $\underline{\varepsilon} \in \Omega^k(\mathbf{R}^n)$.

☞ Proof. Let us prove that j_{2n}^* is injective. Let $x = (x_i)_{i=1}^n \in \mathbf{R}^n$ and $\iota_n\colon x \mapsto (x_i, 0)_{i=1}^n$ from $\mathbf{R}^n$ into $\mathbf{R}^{2n}$. Let $j_n\colon \mathbf{R}^n/\{\pm1\}^n \to K^n$ be defined by $j_n\colon \mathrm{class}(x) \mapsto \mathrm{sq}(x) = (x_i^2)_{i=1}^n$. Then, $j_n = j_{2n} \circ \underline{\iota}$, where $\underline{\iota}$ is the projection of ι, from $\mathbf{R}^n/\{\pm1\}^n$ to $\mathcal{Q}^n$.

$$\begin{array}{ccccc}
\mathbf{R}^n & \xrightarrow{\quad\iota\quad} & \mathbf{R}^{2n} & & \\
\Big\downarrow{\scriptstyle \mathrm{class}} & & \Big\downarrow{\scriptstyle \mathrm{class}} & & \\
\mathbf{R}^n/\{\pm1\}^n & \xrightarrow[\underline{\iota}]{} & \mathcal{Q}_n = \mathbf{R}^{2n}/\mathrm{SO}(2,\mathbf{R})^n & \xrightarrow[j_{2n}]{} & K^n = [0,\infty[^n
\end{array}$$

But $\mathrm{sq} = j_n \circ \mathrm{class}$ and we know that $\mathrm{sq}^* = \mathrm{class}^* \circ j_n^*\colon \Omega^k(K^n) \to \Omega^k(\mathbf{R}^n)$ is injective (art. 258), thus $j_n^*\colon \Omega^k(K^n) \to \Omega^k(\mathbf{R}^n/\{\pm1\})$ is injective. On the other hand, $j_n = j_{2n} \circ \underline{\iota}$, then $j_n^* = \underline{\iota}^* \circ j_{2n}^*$. Since j_n is injective, j_{2n}^* is necessarily injective too. ▶



[6] Which is not a diffeomorphism [PIZ07].

Differential Forms on the Cross

We describe the differential forms on the diffeological space consisting of the union of the two axes $ox \cup oy$ in $\mathbf{R}^2$.

This question was raised by professor Jedrzej Sniatycki: describe the differential forms on the space X, union of the two axes in $\mathbf{R}^2$

$$X = \{(x,0) \mid x \in \mathbf{R}\} \cup \{(0,y) \mid y \in \mathbf{R}\},$$

equipped with the subset diffeology.

261. The forms on the cross. Every differential 1-form α on X, is the restriction of a 1-form on $\mathbf{R}^2$ of type $A = a(x)dx + b(y)dy$. In other words, for all plots $P\colon r \mapsto (x_r, y_r)$ in X,

$$\alpha(P)_r(\delta r) = a(x_r)\frac{\partial x_r}{\partial r}(\delta r) + b(y_r)\frac{\partial y_r}{\partial r}(\delta r).$$

The 1-forms a and b are the restrictions of α on the axes:

$$a(x)dx = \alpha \restriction ox \quad \text{and} \quad b(y)dy = \alpha \restriction oy.$$

Note. According to the definition [**TB**, § 6.40], the value of α at $(0,0)$ is zero if and only if $a(0) = b(0) = 0$. Indeed evaluated on the plots $t \mapsto (t,0)$, and $t \mapsto (0,t)$ at $t = 0$, α is equal to $a(0)dx$ or $b(0)dy$, which are not zero, except if both are 0.

Proof. First of all, notice that the four semi-axes: ox^-, ox^+, oy^- and oy^+, are D-open in X. That is, open for the D-topology [**TB**, § 2.8]. Indeed, for example, for any plot P in X, $P^{-1}(ox^+) =$

$P^{-1}\{(x,y) \mid x > 0\}$, which is open because it is the pullback of an open subset by a continuous map. And *mutatis mutandis* for ox^-, oy^- and oy^+.

Let P be defined on $U \subset \mathbf{R}^n$, and let

$$\mathcal{O} = P^{-1}(ox^- \cup ox^+ \cup oy^- \cup oy^+).$$

The subset $\mathcal{O}$ is open. Let us prove first that,

$$\alpha - A \restriction \mathcal{O} = 0.$$

That is, for all $r \in \mathcal{O}$ and all $\delta r \in \mathbf{R}^n$

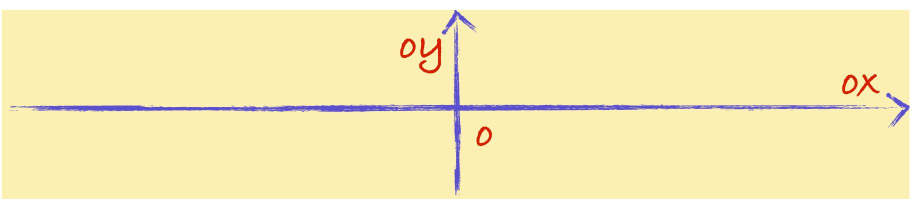
$$\alpha(P)_r(\delta r) - a(x_r)\delta x_r - b(y_r)\delta y_r = 0.$$

We denoted

$$\delta x_r = \frac{\partial x_r}{\partial r}(\delta r) \quad \text{and} \quad \delta y_r = \frac{\partial y_r}{\partial r}(\delta r).$$

Let $r \in \mathcal{O}$ and $P(r) \in ox^+$, for example. Then, there is an open neighbourhood V of r such that $P \restriction V$ takes its values in ox^+. Thus, on V, by definition $\alpha(P)_r(\delta r) = a(x_r)\delta x_r$. And since $y_r = 0$ on V, $\alpha(P)_r(\delta r) = a(x_r)\delta x_r + b(y_r)\delta y_r$. *Mutatis mutandis* for all other values of $P(r)$ when $r \in \mathcal{O}$. We get eventually that on $\mathcal{O}$, $(\alpha - A)(P) \restriction \mathcal{O} = 0$.

Now, for all $\delta r \in \mathbf{R}^n$, $(\alpha - A)(P)_r(\delta r)$ is a smooth function in r. This function vanishes on $\mathcal{O}$, by continuity it vanishes on the closure $\overline{\mathcal{O}}$. Let $U' = U - \overline{\mathcal{O}}$, U' is an open subset of $\mathbf{R}^n$ and then $P \restriction U'$ is a plot in X. Then, it makes sense to evaluate $(\alpha - A)(P)_r(\delta r)$ on U'. But since, for all $r \in U'$, $P(r) = (0,0)$, $(\alpha - A)(P)_r(\delta r) = 0$. Indeed, a differential form evaluated on a constant plot is zero (since it factorizes through $\mathbf{R}^0$). Thus, for all $r \in U$, $(\alpha - A)(P)_r(\delta r) = 0$. Hence $\alpha(P)_r(\delta r) = a(x_r)\delta x_r + b(y_r)\delta y_r$, and $\alpha = A \restriction X$. ▶



A note on Hamiltonian Diffeomorphisms

We describe the group $\mathrm{Ham}(X, \omega)$, of Hamiltonian transformations of a parasymplectic space (X, ω), as the kernel of a morphism into the product of the tori of periods of the elements of the holonomy groups.

We consider a diffeological space X equipped with a *parasymplectic* form ω, that is, a closed 2-form on X. Let G_ω be the group $\mathrm{Diff}(X, \omega)$ and G_ω° be its identity component. Let $\tilde{G}_\omega^\circ$ be the universal covering of G_ω°, and $\pi : \tilde{G}_\omega^\circ \to G_\omega^\circ$ be the projection. Let Γ_ω be the holonomy of ω, defined in [TB, § 9.7], that is,

$$\Gamma_\omega = \{\Psi_\omega(\ell) \mid \ell \in \mathrm{Loops}(X)\},$$

where Ψ_ω is the universal moment map, *i.e.* the moment map relative to the group G_ω. Let us remind that Γ_ω is made of closed 1-forms on G_ω. Then, for all $\gamma \in \Gamma_\omega$, there exists a unique homomorphism $\bar{\gamma} : \tilde{G}_\omega^\circ \to \mathbf{R}$ such that $d\bar{\gamma} = \pi^*(\gamma)$. Let

$$\hat{H}_\omega = \bigcap_{\gamma \in \Gamma_\omega} \ker(\bar{\gamma}).$$

The group of Hamiltonian transformations is then defined (op. cit. § 9.15) by

$$\mathrm{Ham}(X, \omega) = \pi(\hat{H}_\omega).$$

That is what is written in the book *Diffeology* in the sections 9.15 and 9.16. But we can go a little bit further and write the group $\mathrm{Ham}(X, \omega)$

directly as the kernel of one homomorphism. For all $\gamma \in \Gamma_\omega$, let P_γ the group of periods:

$$P_\gamma = \left\{ \int_\ell \gamma \mid \ell \in \mathrm{Loops}(G_\omega, 1) \right\}.$$

We assume now that the periods P_γ is a strict subgroup of $\mathbf{R}$. The fonction $\bar{\gamma}$ projects onto a smooth homomorphism $\underline{\gamma}$ from G_ω° into the torus of periods

$$T_\gamma = \mathbf{R}/P_\gamma,$$

and the projection class : $\mathbf{R} \to T_\gamma$ is the universal covering. This situation is illustrated by the following commutative diagram of homomorphisms.

$$\begin{array}{ccc} \tilde{G}_\omega^\circ & \xrightarrow{\ \bar{\gamma}\ } & \mathbf{R} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \mathrm{class}} \\ G_\omega^\circ & \xrightarrow[\ \underline{\gamma}\ ]{} & T_\gamma \end{array}$$

<u>**262. The group of Hamiltonian diffeomorphisms.**</u> Let $\gamma \in \Gamma_\omega$. The projection of $\ker(\bar{\gamma}) \subset \tilde{G}_\omega^\circ$ in G_ω°, by π, is the kernel of $\underline{\gamma}$. Therefore,

$$\mathrm{Ham}(X, \omega) = \bigcap_{\gamma \in \Gamma_\omega} \ker(\underline{\gamma}).$$

Defining

$$\eta : G_\omega^\circ \to \prod_{\gamma \in \Gamma_\omega} T_\gamma \quad \text{by} \quad \eta(g) = (\underline{\gamma}(g))_{\gamma \in \Gamma_\omega},$$

we have then

$$\mathrm{Ham}(X, \omega) = \ker(\eta).$$

Proof. Let $g = \pi(\tilde{g})$ and $\tilde{g} \in \ker(\bar{\gamma})$, then $\underline{\gamma}(g) = \underline{\gamma}(\pi(\tilde{g})) = \mathrm{class}(\bar{\gamma}(\tilde{g})) = \mathrm{class}(0) = 1$. We denote the group operation on T_γ multiplicatively. Thus, $\pi(\ker(\bar{\gamma})) \subset \ker(\underline{\gamma})$. Now, let $g \in \ker(\underline{\gamma})$ and let $\tilde{g} \in \pi^{-1}(g)$. Thus $\bar{\gamma}(\tilde{g}) \in P_\gamma$. But since P_γ is the group of periods of γ, there exists a loop ℓ in G_ω° such that

$$\bar{\gamma}(\tilde{g}) = \int_\ell \gamma.$$

On the other hand, $\tilde{g}$ is the fixed-ends homotopy class of a path p in G_ω° pointed at the identity, and

$$\bar{\gamma}(\tilde{g}) = \int_p \gamma.$$

Next, let $\tilde{g}' = \bar{\ell} \vee p$, where $\bar{\ell}$ is the reverse loop $\bar{\ell}(t) = \ell(1-t)$. Technically, we work with stationary paths and the concatenation is an internal operation. Then,

$$\pi(\tilde{g}') = g \quad \text{and} \quad \bar{\gamma}(\tilde{g}') = \int_{\bar{\ell}\vee p} \gamma = \int_{\bar{\ell}} \gamma + \int_p \gamma = -\int_\ell \gamma + \int_p \gamma = 0.$$

Thus, $g \in \pi(\ker(\bar{\gamma}))$. Therefore, $\pi(\ker(\bar{\gamma})) = \ker(\underline{\gamma})$. ▶

The next proposition seems to be obvious but needs a proof regarding the diffeology involved (op. cit. § 6.29).

263. $\{0\} \subset \Omega^k(X)$ is closed. The subset $\{0\} \subset \Omega^k(X)$ is closed for the D-topology of the functional diffeology of $\Omega^k(X)$.

Proof. Let us remind that a subset of a diffeological space is open if and only if its pullback by any plot is open (op. cit. § 2.8). Alternatively, a subset is closed if and only if its pullback by any plot is closed. So, let $P : r \mapsto \alpha_r$ a plot defined on U in $\Omega^k(X)$. Let

$$\begin{aligned} P^{-1}(0) &= \{r \in U \mid \alpha_r = 0\} \\ &= \{r \in U \mid \text{For all plots in } X, \alpha_r(Q)_s = 0\} \\ &= \bigcap_{Q \in \mathcal{D}} \{r \in U \mid \text{For all } s \in V, \alpha_r(Q)_s = 0\} \end{aligned}$$

where Q is defined on some domain V, $s \in V$ and $\alpha_r(Q)_s \in \Lambda^k(\mathbf{R}^n)$, the vector space of k-linear forms on $\mathbf{R}^n$, assumes P is an n-plot. Let

$$PQ = \{(r,s) \in U \times V \mid \alpha_r(Q)_s = 0\}.$$

Since $(r,s) \mapsto \alpha_r(Q)_s$ is smooth, then continuous, the subset PQ is closed. Now,

$$P^{-1}(0) = \bigcap_{Q \in \mathcal{D}} \mathrm{pr}_1(PQ),$$

where $\mathrm{pr}_1 : (r,s) \mapsto r$ is an open map. Thus, since PQ is closed then $\mathrm{pr}_1(PQ)$ is closed and $P^{-1}(0)$ is an intersection of closed subsets, hence closed. Therefore, $P^{-1}(0)$ is closed for every plot P in $\Omega^k(X)$, that is, $\{0\} \subset \Omega^k(X)$ is closed. ▶

<u>**264. The D-topology of** $\mathrm{Diff}(\mathbf{X}, \omega)$.</u> Let (X, ω) be a parasymplectic space. For the D-topology, $\mathrm{Diff}(X, \omega)$ is closed in $\mathrm{Diff}(X)$.

Proof. Let us consider the map

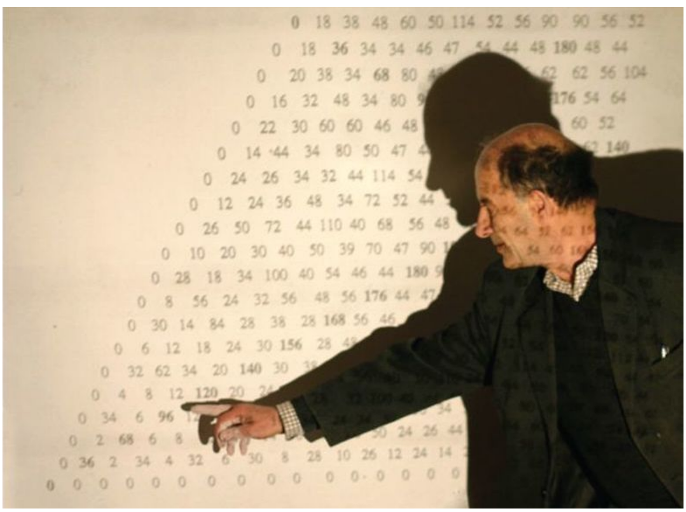
$$F : \mathrm{Diff}(X) \to \Omega^2(X) \quad \text{with} \quad F : \varphi \mapsto \varphi^*(\omega) - \omega.$$

The group $\mathrm{Diff}(X, \omega)$ is the kernel of F, that is,

$$\mathrm{Diff}(X, \omega) = \ker(F) = F^{-1}(0).$$

Since the pullback is a smooth operation (op. cit. § 6.32), with $\Omega^2(X)$ equipped with the functionnal diffeology, and since $0 \in \Omega^2(X)$ is closed for the D-topology, we checked that just above, $\mathrm{Diff}(X, \omega)$ is closed in $\mathrm{Diff}(X)$. ▶

Clearly this proposition applies to any group $\mathrm{Diff}(X, \alpha)$ of automorphisms, where α is any k-form.



Vladimir Igorevich Arnold, a master in Hamiltonian geometry and mechanics.

Differential of a Lie-Group Valued Function

In this note we give a meaning of the differential of a smooth function defined on a diffeological space, with values in a Lie group.

66. The general question

This question has been raised by Cheyne Miller,[1] as a byproduct of his reflexion on the differential of the holonomy function, in the case of a general fiber bundle, or a more general situation. I am sure he will not see any objection that I share my reflexion on his question in these notes.

Precisely, a part of his question translates into:

Question (Cheyne Miller) *Let $h : \mathrm{X} \to \mathrm{G}$ be a smooth function defined on a general diffeological space, with values into an ordinary Lie group. What would be the meaning of dh ?*

When G is the abelian group $(\mathbf{R}, +)$, we know the answer:

$$dh = h^*(dt),$$

where dt is the standard 1-form on $\mathbf{R}$. More recently we have seen, in the previous note "Differential of holonomy For torus bundles", that we can define a "differential" $d_{\mathrm{T}}h$ by:

$$d_{\mathrm{T}}h = h^*(\theta),$$

[1] Private email exchange of the February 27, 2016.

where θ is the standard 1-form on the (maybe irrational) torus $T = \mathbf{R}/\Gamma$, pushforward of dt by the projection $\pi : \mathbf{R} \to T$. In other words, satisfying $\pi^*(\theta) = dt$.

We remark that, in these two cases, the differential of the function is the pullback of some fundamental form defined on the group $\mathbf{R}$ or T. This consideration will be the guide to define more generally the differential of a function with values in any Lie group.

67. The Maurer-Cartan form

There is a canonical 1-form defined on any Lie group G, with values on the tangent space $\mathcal{G} = T_1 G$, that is, the Maurer-Cartan form $\Theta \in \Omega^1(G, \mathcal{G})$. With the ordinary notations of differential calculus on a Lie group, let $g \in G$, let $L(g)$ be the left-multiplication by g, that is, $L(g)(g') = gg'$. Let $\delta g \in T_g G$, be any tangent vector at the point g, then the Maurer-Cartan form Θ is defined by[2]

$$\Theta_g(\delta g) = [D(L(g))(1)]^{-1}(\delta g).$$

This Maurer-Cartan form is left-invariant, that is, $L(k)^*(\Theta) = \Theta$ for all $k \in G$. We have an analogous right-invariant Maurer-Cartan form, replacing $L(g)$ by $R(g)$ in the definition above. The choice depends on our need or preferences.

Note that the Maurer-Cartan form has a diffeology-compliant definition. Let $P : r \mapsto g_r$ be a n-plot of G, let $v \in \mathbf{R}^n$, then:

$$\Theta(P)_r(v) = D(s \mapsto g_r^{-1} g_{r+s})(s = 0)(v).$$

68. The definition of the differential

Now we have the tools to define the differential of the smooth function $h : X \to G$. I suggest this:

265. Definition. *Let X be a diffeological space, let G be an ordinary Lie group, let $h : X \to G$ be a smooth function. We define the*

[2]The notation $D(f)(x)$ means: the tangent linear map of f at the point x.

differential *of h as the pullback by h of the Maurer-Cartan form. And we shall denote,*

$$d_{\mathrm{G}}h = h^*(\theta), \quad d_{\mathrm{G}}h \in \Omega^1(\mathrm{X}, \mathcal{G}).$$

This is a well defined diffeological 1-form with values in a vector space, and that corresponds exactly to what we did in the particular abelian case of $\mathbf{R}$ or T. In that case the Maurer-Cartan form is just the identity.

Now, how is that a good replacement for the differential in the general case? If X is a manifold, we should have on the one hand, for all $x \in \mathrm{X}$ and $\delta x \in \mathrm{T}_x\mathrm{X}$:

$$dh_x(\delta x) \in \mathrm{T}_{h(x)}\mathrm{G},$$

where dh_x, also denoted by $\mathrm{D}(h)(x)$, is the tangent linear map of h at the point x. The Maurer-Cartan form is just the tool for trivialising the tangent fiber bundle. The map $(g, v) \mapsto (g, \theta_g(v))$ is the canonical isomorphism from TG to $\mathrm{G} \times \mathcal{G}$. Therefore $\theta_{h(x)}(dh_x(\delta x))$ is the image of δx by dh_x after trivialisation.

On the other hand, $\theta_{h(x)}(dh_x(\delta x))$ is, by definition, $h^*(\theta)_x(\delta x)$. Therefore, the definition given above $d_{\mathrm{G}}h = h^*(\theta)$ seems to be a good replacement, in diffeology, of the differential of a Lie-group valuated smooth function.

Note that in the 1-dimension case $\mathrm{G} = \mathbf{R}$ or T the differential $d_{\mathrm{G}}h$ is closed (because 2-forms on 1-dimensional spaces vanishe), which is no more true in the general case. Indeed, $d[d_{\mathrm{G}}h] = h^*(d\theta) \in \Omega^2(\mathrm{X}, \mathcal{G})$. The differential d_{G} is a replacement for the tangent linear map, not for the differential operator of exterior calculus. But this is what is useful in mathematical physics when one deals with principal connections, or more exotic structures like gerbes.

$$\Theta(\mathrm{P})_r(v) = \mathrm{D}(s \mapsto g_r^{-1}\, g_{r+s})(s=0)(v)$$

The Geodesics of the 2-Torus

We describe the space of geodesics trajectories of the 2-torus, and we exhibit its natural parasymplectic structure.

It is well known that, if the space of (oriented) *geodesic trajectories* (*a.k.a.* unparamatrized geodesics) of a manifold is a manifold, then this manifold is naturally symplectic. A famous example is the geodesics of the sphere S^2, for which its space of geodesics is also S^2, equipped with the standard surface element.[1] The mapping from the unit bundle $US^2 = \{(x, u) \in S^2 \times S^2 \mid u \cdot x = 0\}$ to $\mathrm{Geod}(S^2)$ is realized by the moment map $\ell : (x, u) \mapsto x \wedge u$. Now,

Question What about the space of geodesics of the torus T^2 ?

It is certainly not a manifold because of the mix of closed and not closed geodesics. And what about the canonical symplectic structure, does it remain something from it? And what?

In the following, we denote by $\mathrm{pr} : \mathbf{R}^2 \to T^2 = \mathbf{R}^2/\mathbf{Z}^2$ the projection.

As usual in symplectic geometry, we realize the space $\mathrm{Geod}(T^2)$ as the space of characteristics of the presymplectic form $d\lambda$ on the unit bundle $UT^2 \simeq T^2 \times S^1$, where

$$\lambda(\delta y) = u \cdot \delta x,$$

[1] for a judicious choice of constant.

with $y = (\mathrm{pr}(x), u) \in \mathrm{UT}^2$ and $\delta y = (\mathrm{D}(\mathrm{pr})_x(\delta x), \delta u) \in \mathrm{T}_y\mathrm{UT}^2$. Thus, the characteristics of $d\lambda$ are the submanifolds

$$\gamma = \Big(\mathrm{pr}\{x + tu\}_{t\in\mathbf{R}}, u\Big),$$

when $(\mathrm{pr}(x), u)$ is a point in UT^2. We denote by π the projection from UT^2 to $\mathrm{Geod}(\mathrm{T}^2)$,

$$\pi : (\mathrm{pr}(x), u) \mapsto \Big(\mathrm{pr}\{x + tu\}_{t\in\mathbf{R}}, u\Big).$$

The space $\mathrm{Geod}(\mathrm{T}^2)$ of unparametrized geodesics of the 2-Torus is naturally a bundle[2] over the circle S^1, thanks to the projection

$$\mathrm{pr}_2 : \Big(\mathrm{pr}\{x + tu\}_{t\in\mathbf{R}}, u\Big) \mapsto u.$$

The preimage $\mathrm{Geod}_u(\mathrm{T}^2) = \mathrm{pr}_2^{-1}(u)$ is the space of all the geodesics with slope u, that is, the torus $\mathrm{T}_u = \mathrm{T}^2/\Delta_u$, where $\Delta_u = \mathrm{pr}(\mathbf{R}u)$. Precisely, T_u is a *rational torus* (a circle) if u is a rational vector, that is, if $\mathbf{R}u \cap \mathbf{Z}^2 \simeq \mathbf{Z}$. And T_u is an *irational torus* [DI83] if $\mathbf{R}u \cap \mathbf{Z}^2 = \{0\}$.

$$\begin{array}{ccc}
\mathrm{U}\mathbf{R}^2 & \xrightarrow{\ \pi\ } & \mathrm{Geod}(\mathbf{R}^2) \\
\downarrow & & \downarrow \\
\mathrm{UT}^2 & \xrightarrow{\ \pi\ } & \mathrm{Geod}(\mathrm{T}^2) \\
& {\scriptstyle \mathrm{pr}_2}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_2} & \\
& \mathrm{S}^1 &
\end{array}$$

266. Parasymplectic structure on $\mathrm{Geod}(\mathrm{T}^2)$. There exists a closed 2-form ω on $\mathrm{Geod}(\mathrm{T}^2)$ such that $\pi^*(\omega) = d\lambda$. We say that ω is parasymplectic.[3]

[2] But not a fiber-bundle.

[3] The analysis of $\mathrm{Diff}(\mathrm{Geod}(\mathrm{T}^2), \omega)$ and the universal moment map [TB, § 9.14] is discussed in "Diffeomorphisms of $\mathrm{Geod}(\mathrm{T}^2)$", in these notes.

Note. It is noteworthy that, in spite of the singularities of the space of geodesics, some closed and diffeomorphic to the circle (indexed by $\mathbf{Q}$) and the others unclosed and diffeomorphic to $\mathbf{R}$, the presymplectic form $d\lambda$ passes into a smooth closed 2-form on the quotient $\mathrm{Geod}(\mathrm{T}^2)$. And that is the main fact we want to draw your attention to.

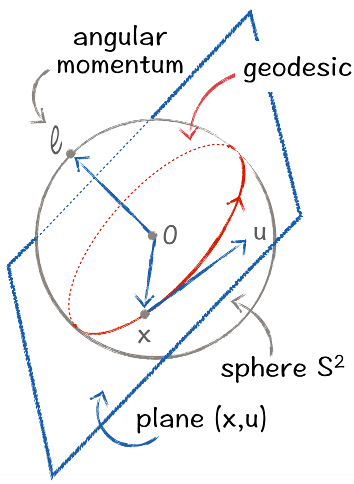
☞ Proof. We use the criterion § 6.38 of [TB] on $d\lambda$. Let $\mathrm{P} : r \mapsto (z_r, u_r)$ and $\mathrm{P}' : r \mapsto (z'_r, u'_r)$ be two plots of UT^2, defined on a same domain and such that $\pi \circ \mathrm{P} = \pi \circ \mathrm{P}'$. That is,

$$\Big(\mathrm{pr}\{x_r + tu_r\}_{t\in\mathbf{R}}, u_r\Big) = \Big(\mathrm{pr}\{x'_r + tu'_r\}_{t\in\mathbf{R}}, u'_r\Big).$$

Thus, $u'_r = u_r$ and $\mathrm{pr}\{x'_r + tu_r\}_{t\in\mathbf{R}} = \mathrm{pr}\{x_r + tu_r\}_{t\in\mathbf{R}}$. Then, $x'_r = x_r + f(r)u_r$, with $f(r) = u_r \cdot (x'_r - x_r)$, f is smooth. Next,

$$\begin{aligned}\lambda(\mathrm{P}')_r(\delta r) &= u_r \cdot \Big(\delta x_r + df_r(\delta r)u_r + f(r)\delta u_r\Big)\\ &= \lambda(\mathrm{P})_r(\delta r) + df_r(\delta r).\end{aligned}$$

That is, $\lambda(\mathrm{P}') = \lambda(\mathrm{P}) + df$. Therefore, $d(\lambda(\mathrm{P}')) = d(\lambda(\mathrm{P}))$. By application of § 6.38 of [TB], there exists $\omega \in \Omega^2(\mathrm{Geod}(\mathrm{T}^2))$ such that $\pi^*(\omega) = d\lambda$ and $d\omega = 0$. ▶



The geodesics of the sphere

The Use of the Moment Map in Geodesic Calculus

A fact or two about geodesics, proved thanks to the moment map.[4]

The moment map is a powerful tool in symplectic geometry. This is another illustration.

69. On geodesics

We consider the unparametrized geodesics of a Riemannian manifold (M, g). They are defined as the characteristics of the presymplectic 2-form $d\lambda$ on the *unit tangent bundle*:

$$\mathrm{UM} = \{(x, u) \in \mathrm{TM} \mid x \in \mathrm{M}, u \in \mathrm{T}_x(\mathrm{M}) \text{ and } u \cdot u = 1\},$$

where TM denotes the tangent bundle and $\lambda \in \Omega^1(\mathrm{UM})$ is defined by its evaluation on a tangent vecteur $\delta(x, u) \in \mathrm{T}_{(x,u)}(\mathrm{UM})$

$$\lambda_{(x,u)}(\delta(x, u)) = u \cdot \delta x.$$

We note $v \cdot w$ for $g_x(v, w)$, and $\|v\|^2 = v \cdot v$, where $v, w \in \mathrm{T}_x(\mathrm{M})$. The characteristics of $d\lambda$ satisfies the differential equations

$$\delta(x, u) \in \ker(d\lambda_{(x,u)}) \quad \text{iff} \quad \delta x \propto u \quad \text{and} \quad \hat{\delta} u = 0,$$

where $\hat{\delta}$ stands for the covariant differentiation. In a chart:

$$\hat{\delta} u^\mu = \delta u^\mu + \Gamma^\mu_{\nu\rho} u^\nu \delta x^\rho,$$

[4]This note comes from a discussion with Jean-Paul Mohsen.

where the $\Gamma^{\mu}_{\nu\rho}$ are the Christoffel symbols of the Levi-Civita connection. The map $\delta(x,v) \mapsto (\delta x, \hat{\delta} v)$ is a isomorphism from $\mathrm{T}_{(x,v)}(\mathrm{TM})$ to $\mathrm{T}_x(\mathrm{M}) \times \mathrm{T}_x(\mathrm{M})$. What we have to know now is this:

267. Geodesics trajectories. *The integral curves of the distribution*

$$(x,u) \mapsto \ker(d\lambda_{(x,u)})$$

projects on M *on the geodesics trajectories (or unparametrized geodesics) of the Riemannian metric g.*

Actually, for every integral curve there exists always a parametrization $t \mapsto x(t)$, defined on some interval, such that $u(t) = dx(t)/dt$, with $(x(t), u(t)) \in \mathrm{UM}$, and $[t \mapsto (x(t), u(t))]$ is a characteristic of $d\lambda$. And that is the way one usually understands the wording "unparametrized geodesics".

Now, if a Lie-group H acts on M preserving the metric g, that is, $h_{\mathrm{M}}^*(g) = g$ for all $h \in \mathrm{H}$, then its natural action on TM preserves UM,

$$h_{\mathrm{UM}}(x,u) = (h_{\mathrm{M}}(x), h_{\mathrm{M}*}(u) = \mathrm{D}(h_{\mathrm{M}})(x)(u)),$$

and also λ on UM,

$$h_{\mathrm{UM}}^*(\lambda) = \lambda.$$

The moment map of this action is the pullback by the orbit map of the 1-form λ [**PIZ10**], that is,

$$\mu(x,u) = [h \mapsto h_{\mathrm{M}}(x,u)]^*(\lambda).$$

Applied on a vector Z in the Lie algebra $\mathcal{H}$ of H, that gives

$$\mu(x,u) \cdot \mathrm{Z} = u \cdot \mathrm{Z}_{\mathrm{M}}(x),$$

where $\mathrm{Z}_{\mathrm{M}}(x)$ denotes the action of Z on M, that is, the infinitesimal action of the 1-parameter group generated by Z.

The **Noether theorem** states that the moment map is constant on the characteristics. That means that if $[t \mapsto x(t)]$ is a geodesic with $u(t) = dx(t)/dt$ unitary, then for all t in the domain of the curve, for all $\mathrm{Z} \in \mathcal{H}$,

$$u(t) \cdot \mathrm{Z}_{\mathrm{M}}(x(t)) = u_0 \cdot \mathrm{Z}_{\mathrm{M}}(x_0),$$

where $(x_0, u_0) = (x(t_0), u(t_0))$, for an arbitrary t_0 in the domain of the curve.

70. Special metrics on principal bundles

Consider now a principal bundle $\pi : M \to B$ with structure group H, equipped with an H-invariant metric g. Consider a connection on M defined by orthogonal projectors: for all $x \in M$ let $Q_x : T_x(M) \to T_x(H_M(x))$ be the vertical projector, and $P_x = \mathbf{1}_x - Q_x$ be the horizontal one. In other words, the horizontal subspace is orthogonal to the fibers. Assume also that the metric g is "calibrated vertically", we mean that there exists a left-invariant metric ε on G such that

$$\hat{x}^*(g) = \varepsilon, \quad i.e. \quad g_x(Z_M(x), Z'_M(x)) = \varepsilon(Z, Z'),$$

where $\hat{x}$ is the orbit map:

$$\hat{x} : H \to M \quad \text{with} \quad \hat{x} : h \mapsto h_M(x).$$

Now, we have two interesting consequences of the conservation of the moment map along the geodesics:

268. Proposition 1. If a geodesic is horizontal at some point, it is horizontal at everypoint.

☞ Proof. To be horizontal in one point means that the unit tangent vector u_0 is orthogonal to the fiber at x_0. Next, being orthogonal to a fiber at t_0 means: $u_0 \cdot Z_M(x_0) = 0$ for all $Z \in \mathcal{H}$, since the vertical tangent space is spaned by the infinitesimal action of H. Then, the moment map being constant along the geodesic , $u(t) \cdot Z_M(x(t)) = 0$ for all t. The whole geodesic is horizontal. ▶

Actually this is a particular case of a more general property:

269. Proposition 1'. Let H be a group of isometries of (M, g). If a geodesic is orthogonal to an orbit of H at one point, it is orthogonal to every orbit it meets.

In particular, if the group has a fix point, every geodesic emerging from this point is orthogonal to every orbit it meets. Consider for example the 2-sphere, and the rotations around an axe. The poles are fixed and the geodesics passing through the poles are the meridians, orthogonal to the parallels, orbits of the rotations.

270. **Proposition 2.** The fibers are totally geodesic.

Proof. Let us recall that being totally geodesic means that a geodesic which is tangent to a fiber at some point is tangent to that fiber at every point.

Let $\{Z_i\}_{i=1}^k$ be an orthonormal basis of $\mathcal{H}$, for the metric ε. Then, according to the hypothesis $\hat{x}^*(g) = \varepsilon$, the vectors $Z_{iM}(x)$ form an orthonormal basis of $T_x(H_M(x))$, for all $x \in M$. Let $t \mapsto x(t)$ be a geodesic with unitary velocity $u(t)$, and let $u(t) = u_Q(t) + u_P(t)$, where u_Q is the vertical part and u_P the horizontal. Then, $\|u_Q(t)\|^2 = \sum_{i=1}^k u_i(t)^2$, with $u_i(t) = u(t) \cdot Z_{iM}(x(t))$. Thus, since the $u_i(t)$ are constant on the geodesic, $\|u_Q(t)\|^2$ is also constant on the geodesic. Thus, if $u_P(x_0) = 0$, then $\|u_Q(t)\|^2 = \|u_Q(0)\|^2 = 1$ for all t, and then $\|u_P(t)\|^2 = 0$. Therefore, if the geodesic is tangent to the fiber somewhere it is tangent everywhere. ▶

271. **Remark.** On $\mathbf{R}^2$ the standard metric is $SO(2, \mathbf{R})$-invariant. And if we forget 0, the action of $SO(2, \mathbf{R})$ is principal. The geodesics are the lines and the orbits, the circles centered at the origin. A line which is tangent to a circle somewhere is not tangent to any other circle. Thus, the orbits are not geodesics. So? Did we miss something? It happens that the metric is not calibrated with respect to the action of $SO(2, \mathbf{R})$. Precisely $\hat{x}^*(g) = \|x\|^2\varepsilon$. Hence, we can see that for the proposition 2, the hypothesis for the metric to be calibrated is crucial.

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\varepsilon} \left\{ \frac{\partial g_{\nu\varepsilon}}{\partial x^{\rho}} + \frac{\partial g_{\rho\varepsilon}}{\partial x^{\nu}} - \frac{\partial g_{\nu\rho}}{\partial x^{\varepsilon}} \right\}$$

The Parasymplectic Space of Geodesics Trajectories

We describe the parasymplectic structure on the space of geodesics trajectories of any Riemannian manifold.

We consider a Hausdorff and second countable manifold M equipped with a Riemannian metric g. The construction of the space of geodesic trajectories is described in §69. Quotient spaces are equipped with quotient diffeology. Let UM be the unit tangent bundle

$$\mathrm{UM} = \{y = (x, u) \mid x \in \mathrm{M} \text{ and } u \in \mathrm{T}_x\mathrm{M} \text{ with } g(u, u) = 1\}.$$

The space of (oriented) geodesic trajectories Geod(M) is defined as the space of characteristics of the presymplectic form $d\lambda$ defined on UM, where λ is the Cartan form

$$\lambda(\delta y) = g(u, \delta x),$$

for all $\delta y \in \mathrm{T}_y\mathrm{UM}$. Its differential $d\lambda$ is given by

$$d\lambda(\delta y, \delta' y) = g(\hat{\delta} u, \delta' x) - g(\hat{\delta}' u, \delta x),$$

where $\hat{\delta} u$ is defined by its coordinates in the charts:

$$\hat{\delta} u^\mu = \delta u^\mu + \Gamma^\mu_{\nu\rho} u^\nu \delta x^\rho.$$

The $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols. We have seen that $\delta y \in \ker(d\lambda)$ if and only if:

$$\hat{\delta} u = 0 \text{ and } \delta x \propto u.$$

Thus, there is only one vector $\xi \in \mathrm{T}_y\mathrm{UM}$ such that:

$$\xi \in \ker(d\lambda) \text{ and } \lambda(\xi) = 1.$$

It is defined by

$$\xi = \delta y \text{ with } \delta x = u \text{ and } \hat{\delta} u = 0.$$

This vector ξ is called *Reeb vector*.

Definition. *A* contact form *on a manifold* Y *is any* 1*-form* λ *such that*

$$\ker(\lambda) \cap \ker(d\lambda) = \{0\},$$

and the Reeb vector field *is defined by*

$$\xi \in \ker(d\lambda) \text{ and } \lambda(\xi) = 1.$$

We accept th following proposition:

Proposition. *Two point y and y' are on the same characteristic of $d\lambda$, that is, on an integral curve of the distribution $y \mapsto \ker(d\lambda)$, if and only if there is a real number s such that:*

$$e^{s\xi}(y) = y',$$

where $e^{s\xi}$ denotes the local flow $e^{t\xi}$ of the Reeb vector ξ.

The Reeb vector field on UM generates what is called the *geodesic flow*. We have then the following theorem:

Theorem. *Let Y be a manifold, Hausdorff and second countable, equipped with a contact form λ. Then, there always exists on the space $\mathcal{S}$ of charateristics of $d\lambda$ a parasymplectic form ω such that*

$$\text{class}^*(\omega) = d\lambda,$$

where class : $Y \to \mathcal{S} = Y/\ker(d\lambda)$ *is the projection. When $\mathcal{S}$ is a manifold then ω is symplectic, but in any case ω still is a closed* 2*-form on $\mathcal{S}$. This is an example of* symplectic reduction *with singularities similar to the one in Ex. 227. Moreover, if Y has dimension $2n-1$ (contact manifolds are odd dimensional) then $\mathcal{S}$ has dimension $2n-2$.*

Proof. We recall that the Reeb vector field of a contact form generates by integration a local 1-parameter of automorphisms of λ. Indeed, thanks to the Cartan formula one has first the vanishing Lie derivative:

$$\mathcal{L}_\xi(\lambda) = [d\lambda](\xi) + d[\lambda(\xi)] = 0 + 0.$$

Thus, the local flow $e^{t\xi}$ of the vector ξ preserves λ:

$$e^{t\xi *}(\lambda) = \lambda.$$

To prove that the form $d\lambda$ passes to the quotient X, we shall apply the criterion [**TB**, §6.38]. Let $\mathrm{P}, \mathrm{P}' \colon \mathrm{U} \to \mathrm{Y}$ be two plots in Y such that $\pi \circ \mathrm{P} = \pi \circ \mathrm{P}'$. Let $r_0 \in \mathrm{U}$, by hypothesis, $\mathrm{P}(r_0)$ and $\mathrm{P}'(r_0)$ belongs to the same characteristic. Then, there exists a real number s such that $e^{s\xi}(\mathrm{P}(r_0)) = \mathrm{P}'(r_0)$. The local diffeomorphism $e^{s\xi}$ is defined on some neighbourhood of $\mathrm{P}(r_0)$ and its inverse $e^{-s\xi}$, on some neighborhood of $\mathrm{P}'(r_0)$. The composite $\mathrm{P}'' = e^{-s\xi} \circ \mathrm{P}'$ is then defined on some neighbourhood $\mathcal{O}'$ of r_0 and satisfies: $\pi \circ \mathrm{P} = \pi \circ \mathrm{P}''$ and $\mathrm{P}(r_0) = \mathrm{P}''(r_0)$. Now, thanks to the *flow box theorem*, there exists a local diffeomorphism φ from a neighbourhood $\mathcal{O}''$ of $\mathrm{P}(r_0)$, that maps $\mathcal{O}''$ to a product $\mathrm{W} \times \mathcal{I}'$, where $\mathcal{I}'$ is some real interval, the vector field ξ to $(0,1)$ and $e^{s\xi}$ to the translation $(w,t) \mapsto (w, t+s)$. We assume that $\varphi(\mathrm{P}(r_0)) = (0,0)$. The intersection $\mathrm{U}' = (\varphi \circ \mathrm{P})^{-1}(\mathrm{W} \times \mathcal{I}') \cap (\varphi \circ \mathrm{P}'')^{-1}(\mathrm{W} \times \mathcal{I}')$ is open and non empty since it contains r_0. We will restrict P and P$'$ to U$'$. Consider now, $(\pi \circ \varphi^{-1}) \restriction \mathrm{W} \times \{0\}$, since our manifolds are assumed Hausdorff and second countable, the preimages in W of characteristics are countable, and because W is an Euclidean domain, they are (diffeologically) discrete [**TB**, Exercice 8]. Therefore, restricting P and P$''$ to a ball $\mathcal{B}$ centered at r_0, $\pi \circ \mathrm{P} = \pi \circ \mathrm{P}''$ implies $\mathrm{pr}_1 \circ \varphi \circ \mathrm{P}(r) = \mathrm{pr}_1 \circ \varphi \circ \mathrm{P}''(r)$, for all $r \in \mathcal{B}$. That is, $\varphi \circ \mathrm{P}(r) = (f(r), t(r))$ and $\varphi \circ \mathrm{P}''(r) = (f(r), t''(r))$, where the maps involved are smooth. Hence, there a smooth real map $r \mapsto s(r) = t''(r) - t(r)$ such that $\mathrm{P}''(r) = e^{s(r)\xi}(\mathrm{P}(r))$, for all $r \in \mathcal{B}$. Let us write simply $\underline{s(r)}_{\mathrm{Y}} = e^{s(r)\xi}$, then $\mathrm{P}''(r) = \underline{s(r)}_{\mathrm{Y}}(\mathrm{P}(r))$, and $s \mapsto \underline{s}_{\mathrm{Y}}$ is a local additive action on Y. Remember now that because $\xi \in \ker(d\lambda)$ and $\lambda(\xi) = 1$ the Lie derivative vanishes: $\mathcal{L}_\xi(\lambda) = 0$, and then $(e^{s\xi})^*(\lambda) = \lambda$, or $\underline{s}_{\mathrm{Y}}^*(\lambda) = \lambda$. Thus, thanks to [**TB**, §8.37],

$$\begin{aligned}
\lambda(\mathrm{P}'')_r &= \lambda\big(r \mapsto \underline{s(r)}_{\mathrm{Y}}(\mathrm{P}(r))\big)_r \\
&= [\underline{s(r)}_{\mathrm{Y}}^*(\lambda)](\mathrm{P})_r + [\mathrm{R}(\mathrm{P}(r))^*(\lambda)](s)_r \\
&= \lambda(\mathrm{P})_r + ds_r,
\end{aligned}$$

where we denoted $R(y)(s) = \underline{s}_Y(y)$. Hence, $d\lambda(P'') = d\lambda(P)$ on the ball $\mathcal{B}$, and then on all U. Therefore, according to the criterion [**TB**, §6.38], there exists a 2-form ω on X such that $d\lambda = \pi^*(\omega)$. And ω is closed since $\pi^*(d\omega) = 0$ [**TB**, §6.39].

As for the dimension of the quotient, the diffeology of $\mathcal{S} = Y/\ker(d\lambda)$ is generated by the transversal of the flow of the Reeb vector which are $2n-2$ plots. When one can choose such transversal that cuts each orbit of the geodesic flow in one and only one point, then they are local diffeomorphisms with the quotient $\mathcal{S}$, and $\mathcal{S}$ is a manifold. On that manifold the 2-form ω is non degenerate, therefore symplectic. Otherwise, and this is the most general case, one cannot find such transversal everywhere and the flow cuts some of them infinitely many times. It is what happens in particular on the 2-torus for irrational geodesics. ▶

Corollary. *The space* Geod(M) *of geodesic trajectories, equipped with the quotient diffeology of the unit tangent bundle* UM *by the kernel of the differential $d\lambda$, has dimension $2n-2$ and is parasymplectic for a closed 2-form ω defined by* $\text{class}^*(\omega) = d\lambda$. *When* Geod(M) *is a manifold then ω is symplectic.*

Definition. *We shall call* symplectically generated *any parasymplectic space that admits a generating family of plots where the pullback of the parasymplectic form is symplectic.*

Then, the parasymplectic structure on the quotient space $\mathcal{S}$ in the general case of a contact form, and on Geod(M) in particular, is symplectically generated. Indeed, the restrictions of $d\lambda$ on the transversals of the Reeb vector field is non-degenerate since $\ker(d\lambda) = \xi\mathbf{R}$. They give a generating family of plots on which the pullback of ω is symplectic. This is why I decided to call *symplectically generated* this kind of parasymplectic space. That includes the spaces of geodesic trajectories as we just saw, but also the cone orbifolds [**TB**, §9.32] or Prato's toric quasifolds [**EP01**].

Diffeomorphisms of Geod(T^2)

In this note we explicit the group of diffeomorphisms of the space of geodesic trajectories of the 2-torus.

71. Geodesic trajectories of the 2-torus

We consider the space of geodesic trajectories of the 2-torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$, see "The geodesics of the 2-torus" in these notes. They are the projections of the affine lines in $\mathbf{R}^2$ by the projection class : $\mathbf{R}^2 \to T^2$:

$$\text{class} : (x, y) \mapsto \left(e^{2i\pi x}, e^{2i\pi y}\right).$$

272. The geodesics of $\mathbf{R}^2$ The set of (oriented) affine lines in $\mathbf{R}^2$ is diffeomorphic to the tangent space of the circle S^1. That is:

$$\text{Geod}(\mathbf{R}^2) \simeq TS^1 = \left\{(u, r) \in S^1 \times \mathbf{R}^2 \mid u \cdot r = 0\right\}.$$

With every pair (u, r) is associated the line

$$\Delta(u, r) = \{r + tu\}_{t \in \mathbf{R}}.$$

This space is also equivalent to $S^1 \times \mathbf{R}$, thanks to the mapping

$$(u, r) \mapsto (u, \rho = u \cdot Jr), \quad \text{with} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The centered dot denotes the scalar product.

Proof. Contained in Figure 37. ▶

273. The geodesics of T^2 The set of (oriented) affine lines in T^2 is diffeomorphic to the quotient space

$$\mathrm{Geod}(T^2) \simeq (S^1 \times \mathbf{R})/\mathbf{Z}^2,$$

with the $\mathbf{Z}^2$-action defined by

$$\underline{k}(u, \rho) = (u, \rho + u \cdot k) \quad \text{for all} \quad k = (m, n) \in \mathbf{Z}^2.$$

Let $u = (\cos(\theta), \sin(\theta))$, the action of $\mathbf{Z}^2$ on $S^1 \times \mathbf{R}$ writes

$$\underline{(m, n)}(u, \rho) = (u, \rho + m\cos(\theta) + n\sin(\theta)).$$

We denote

$$\mathrm{class}(u, \rho) \equiv (u, \mathrm{class}_u(\rho)) = \big(u, \{\rho + k \cdot u \mid k \in \mathbf{Z}^2\}\big),$$

and

$$\mathrm{pr}_1 : (u, \mathrm{class}_u(\rho)) \mapsto u,$$

the projection of Geod(T^2) to S^1.

274. The projection on the circle The fibers of the projection pr_1 is the torus

$$T_u = \mathbf{R}/[\mathbf{Z}\cos(\theta) + \mathbf{Z}\sin(\theta)].$$

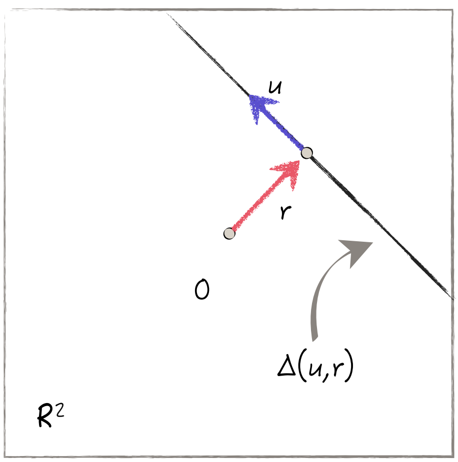


Figure 37. The geodesic trajectories of the plane.

The torus T_u is irrational when $\cos(\theta)$ and $\sin(\theta)$ are independent over $\mathbf{Q}$ and rational, diffeomorphic to a circle $\mathbf{R}/a\mathbf{Z}$, otherwise.

It has a group structure:

$$\mathrm{class}_u(\rho) + \mathrm{class}_u(\rho') = \mathrm{class}_u(\rho + \rho').$$

The projection pr_1 has a global section:

$$\sigma \in \mathcal{C}^\infty(S^1, \mathrm{Geod}(T^2)), \quad \sigma : u \mapsto \mathrm{class}_u(0).$$

These are the geodesics trajectories passing through $0 \in \mathbf{R}^2$.

72. Diffeomorphisms of Geod(T^2)

The diffeology of Geod(T^2) is defined as the quotient diffeology of the manifold $S^1 \times \mathbf{R}$ by $\mathbf{Z}^2$, and then

$$\mathrm{Geod}(T^2) \simeq \mathbf{R}^2/\mathbf{Z}^3,$$

where $(\ell, m, n) \in \mathbf{Z}^3$ acts on $\mathbf{R}^2$ by:

$$(\ell, m, n)_{\mathbf{R}^2}(\theta, \rho) = (\theta + 2\pi\ell, x + m\cos(\theta) + n\sin(\theta)).$$

<u>**275. Proposition 1**</u> Let $f \in \mathrm{Diff}(\mathrm{Geod}\, T^2)$, then there exists $M \in \mathrm{GL}(2, \mathbf{Z})$ such that

$$\mathrm{pr}_1 \circ f = \underline{M} \circ f,$$

where $\mathrm{GL}(2,\mathbf{Z})$ acts on S^1 by

$$\underline{M}(u) = u' \quad \Leftrightarrow \quad u' = Mu/\|Mu\|.$$

Let's call Ψ the homomorphism

$$\Psi : \mathrm{Diff}(\mathrm{Geod}(T^2)) \to \mathrm{GL}(2, \mathbf{Z}) \quad \text{with} \quad \Psi(f) = M.$$

We have a short homomorphism sequence:

$$0 \to \ker(\Psi) \to \mathrm{Diff}(\mathrm{Geod}(T^2)) \to \mathrm{GL}(2, \mathbf{Z}) \to 0.$$

☞ Proof. Let $f \in \mathrm{Diff}(\mathrm{Geod}(T^2))$. The plot $f \circ \mathrm{class} \circ \mathrm{class}_1$ has a local smooth lifting Φ, that is,

$$f \circ \mathrm{class} \circ \mathrm{class}_1 =_{\mathrm{loc}} \mathrm{class} \circ \mathrm{class}_1 \circ \Phi.$$

Precisely, if

$$\Phi(\theta, \rho) = (\theta', \rho')$$

then, for all $(\theta, \rho) \in \mathbf{R}^2$ and all $(\ell, m, n) \in \mathbf{Z}^3$ there exists $(\ell', m', n') \in \mathbf{Z}^3$ such that:

$$\Phi(\theta + 2\pi\ell, \rho + m\cos(\theta) + n\sin(\theta)) = (\theta' + 2\pi\ell', \rho' + m'\cos(\theta') + n'\sin(\theta'))$$

Denoting

$$\phi\Big(\operatorname{class}_1(\theta, \rho)\Big) = \operatorname{class}_1\Big(\Phi(\theta, \rho)\Big) \quad \text{with} \quad \operatorname{class}_1(\theta, \rho) = \Big(e^{i\theta}, \rho\Big)$$

we have

$$\operatorname{class}_1 \circ \Phi =_{\text{loc}} \phi \circ \operatorname{class}_1,$$

and therefore

$$\operatorname{class} \circ \phi =_{\text{loc}} f \circ \operatorname{class},$$

$$\begin{array}{ccc}
\mathbf{R} \times \mathbf{R} & \xrightarrow{\Phi} & \mathbf{R} \times \mathbf{R} \\
\Big\downarrow {\scriptstyle \operatorname{class}_1} & & \Big\downarrow {\scriptstyle \operatorname{class}_1} \\
S^1 \times \mathbf{R} & \xrightarrow{\phi} & S^1 \times \mathbf{R} \\
\Big\downarrow {\scriptstyle \operatorname{class}} & & \Big\downarrow {\scriptstyle \operatorname{class}} \\
\operatorname{Geod}(T^2) & \xrightarrow[f]{} & \operatorname{Geod}(T^2)
\end{array}$$

Let $(u, \rho) \in S^1 \times \mathbf{R}$:

$$\phi(u, \rho) = (u', \rho') \quad \text{with} \quad u' = U(u, \rho) \quad \text{and} \quad \rho' = R(u, \rho).$$

For all $k \in \mathbf{Z}^2$, there is $k' \in \mathbf{Z}^2$, depending a priori on k, u, ρ such that:

$$\phi\big(\underline{k}(u, \rho)\big) = \underline{k'}\big(\phi(u, \rho)\big).$$

We denote by $\underline{k}$ the action of $k \in \mathbf{Z}^2$. Thus,

$$\phi(u, \rho + u \cdot k) = \underline{k'}(u', \rho')$$

$$\big(U(u, \rho + k \cdot u), R(u, \rho + k \cdot u)\big) = (u', \rho' + u' \cdot k')$$

$$\big(U(u, \rho + k \cdot u), R(u, \rho + k \cdot u)\big) = \big(U(u, \rho), R(u, \rho) + U(u, \rho) \cdot k'\big).$$

We get:

$$\begin{cases} \mathrm{U}(u, \rho + k \cdot u) = \mathrm{U}(u, \rho) & (\blacklozenge) \\ \mathrm{R}(u, \rho + k \cdot u)\big) = \mathrm{R}(u, \rho) + \mathrm{U}(u, \rho) \cdot k' & (\heartsuit) \end{cases}$$

The first identity ($\blacklozenge$) gives

$$\mathrm{U}(u, \rho) = \mathrm{U}(u, \rho + m\cos(\theta) + n\sin(\theta)),$$

where $u = (\cos(\theta), \sin(\theta))$ and for all $m, n \in \mathbf{Z}$. Then, for all u irrational, by density of $m\cos(\theta) + n\sin(\theta)$ in $\mathbf{R}$, with $m, n \in \mathbf{Z}$, we have:

$$\mathrm{U}(u, \rho) = \mathrm{U}(u, 0).$$

Then, because the set of irrational u is dense in S^1 :

$$\text{For all } u \text{ in } S^1, \quad u' = \mathrm{U}(u)$$

is independant on ρ. Thus, the map ϕ writes

$$\phi : (u, \rho) \mapsto (u' = \mathrm{U}(u), \rho' = \mathrm{R}(u, \rho)).$$

The map f writes then

$$f(u, \mathrm{class}_u(\rho)) = \big(u' = \mathrm{U}(u), \mathrm{class}_{u'}(\mathrm{R}(u, \rho))\big)$$

Thus, f exchange the fibers T_u to $T_{u'}$, and the restriction of f on T_u writes

$$f_u : \mathrm{class}_u(\rho) = \mathrm{class}_{u'}(\rho') \quad \text{with} \quad \rho' = \mathrm{R}(u, \rho).$$

But since f is a diffeomorphism, its restriction to T_u is a diffeomorphisms:

$$f_u = f \upharpoonright T_u \in \mathrm{Diff}(T_u, T_{u'}).$$

There exists then a map $F : S^1 \to S^1$ such that $u' = F(u)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Geod}(T^2) & \xrightarrow{\ f\ } & \mathrm{Geod}(T^2) \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ S^1 & \xrightarrow[\ F\ ]{} & S^1 \end{array}$$

The map F is a diffeomorphism because f is a diffeomorphism and the projection pr_1 is a subduction. Thus, f is an automorphism of the projection pr_1 :

$$[f \mapsto \mathrm{F}] \in \mathrm{Hom}^\infty\big(\mathrm{Diff}(\mathrm{Geod}_{\mathrm{T}^2}), \mathrm{Diff}(\mathrm{S}^1)\big).$$

On the other hand, we now that [DI83] T_u and $\mathrm{T}_{u'}$ are diffeomorphic if and only if:

$$u' = \mathrm{F}(u) \quad \Rightarrow \quad \exists \mathrm{M} \in \mathrm{GL}(2, \mathbf{Z}),\ \mathrm{M}u = \lambda\ u'.$$

Said differently,

$$\exists \mathrm{M} \in \mathrm{GL}(2, \mathbf{Z}),\ \mathbf{R}u' = \mathrm{M}(\mathbf{R}u).$$

A priori M depends on u. Consider the cicle S^1 as the set or direction in $\mathbf{R}^2$, that is,

$$\mathrm{S}^1 = [\mathbf{R}^2 - \{0\}]/]0, \infty[\quad \text{with} \quad \begin{pmatrix} x \\ y \end{pmatrix} \sim \lambda \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{for al} \quad \lambda \in]0, \infty[.$$

The map F has a lifting (at leat locally) on $\mathbf{R}^2 - \{0\}$. Let's call it $\tilde{\mathrm{F}}$:

$$\begin{array}{ccc} \mathbf{R}^2 - \{0\} & \overset{\tilde{\mathrm{F}}}{\dashrightarrow} & \mathbf{R}^2 - \{0\} \\ \text{class}\downarrow & & \downarrow\text{class} \\ \mathrm{S}^1 & \underset{\mathrm{F}}{\longrightarrow} & \mathrm{S}^1 \end{array} \qquad \text{class} \circ \tilde{\mathrm{F}} =_{\mathrm{loc}} \mathrm{F} \circ \text{class}\,.$$

Thus, for all $u \in \mathrm{S}^1$, for all $v = (x, y) \in \mathbf{R}u$

$$\tilde{\mathrm{F}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_u & b_u \\ c_u & d_u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{with} \quad \mathrm{M}_u = \begin{pmatrix} a_u & b_u \\ c_u & d_u \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Z}),$$

and $\tilde{\mathrm{F}}$ is smooth. Let $(x', y') = \tilde{\mathrm{F}}(x, y)$. The map $(x, y) \mapsto x' = a_u x + b_u y$ is smooth. Let $y = 1$ and consider the map $\xi : x \to a_u x + b_u$ were $u = (x, 1)/\sqrt{x^2 + 1}$. Since $\tilde{\mathrm{F}}$ is smooth, ξ is smooth on a neighborhood of 0. Assume $\xi(0) = 0$. Its derivative satisfies

$$\xi'(0) = \lim_{t \to 0} \frac{1}{t}\big(m(t)t + n(t)\big) = \lim_{t \to 0} \left(m(t) + \frac{n(t)}{t}\right),$$

with $m(t) = a_{u(t)}$ and $n(t) = b_{u(t)} - b_{u(0)}$ integers. That is:

$$\lim_{t\to 0}\left(m(t) + \frac{n(t)}{t} - \xi'(0)\right) = 0.$$

Let $t = 1/N$ with $N \in \mathbf{N}$ big. Then,

$$\lim_{N\to\infty}\Big(m(1/N) + N\,n(1/N) - \xi'(0)\Big) = 0,$$

Then $n(t) = b_{u(t)} - b_{u(0)} = 0$ and $m(t) = a_{u(t)} = \xi'(0) = a$. Thus, the matrix M is constant.

Now let us prove that $\Psi : \mathrm{Diff}(\mathrm{Geod}(T^2)) \to \mathrm{GL}(2,\mathbf{Z})$ is surjective. Let $M \in \mathrm{GL}(2,\mathbf{Z})$, consider $\hat{M} : S^1 \times \mathbf{R} \to S^1 \times \mathrm{R}$:

$$\hat{M}(u,\rho) = (u',\rho') \quad \text{with} \quad u' = \frac{Mu}{\|Mu\|} \quad \text{and} \quad \rho' = \frac{\rho}{\|Mu\|}.$$

Then,

$$\hat{M}(u,\rho + k\cdot u) = (u',\rho' + k'\cdot u') \quad \text{with} \quad k' = (M^{-1})^t k.$$

Thus, $\hat{M}$ passes to the quotient $\mathrm{Geod}(T^2) = [S^1 \times \mathbf{R}]/\mathbf{Z}^2$, and $\Psi(\hat{M}) = M$. Therefore Ψ is surjective. ▶

<u>**276. Proposition 2**</u> The identity component of the kernel of the projection $\Psi : \mathrm{Diff}(\mathrm{Geod}(T^2)) \to \mathrm{GL}(2,\mathbf{Z})$ is equivalent to the space of sections of the projection pr_1 :

$$\begin{aligned}\ker(\Psi)^\circ &\simeq \mathcal{S}ec(\mathrm{pr}_1 : \mathrm{Geod}(T^2) \to S^1)\\ &= \{\sigma \in \mathcal{C}^\infty(S^1, \mathrm{Geod}(T^2) \mid \mathrm{pr}_1 \circ \sigma = 1_{S^1}\}\end{aligned}$$

Let $\sigma : u \mapsto \tau_u \in T_u$ be such a section, the diffeomorphism Σ associated with σ is given by addition inside the fiber:

$$\Sigma : (u,\tau) \mapsto (u, \tau + \tau_u).$$

We recall that each fiber $T_u = \mathbf{R}/[\cos(\theta)\mathbf{Z} + \sin(\theta)\mathbf{Z}]$ is an abelian group.

The subgroup $\ker(\Psi) \in \mathrm{Diff}(\mathrm{Geod}(T^2))$ is what is called by physicists the gauge group of the bundle $\mathrm{pr}_1 : \mathrm{Geod}(T^2) \to S^1$. It has actually 2 components. We described the identity component, the second component is obtained my composing with the inversion $(u,\tau) \mapsto (u,-\tau)$.

Proof. Let $f \in \ker(\Psi)$, then $f_u \in \mathrm{Diff}(T_u)$ for all $u \in S^1$. We know that all diffeomorphisms of T_u are the projections of the affine map $x \mapsto \lambda_u x + \mu_u$, where $u \mapsto \mu_u$ is smooth and λ_u is ± 1 or in $\{\pm 1\} \times \mathbf{Z}$ if u is "quadratic", see [DI83]. Thus, since the irrationals non quadratic unit vectors are dense in S^1, $\lambda_u = \pm 1$. Hence, the value $\lambda_u = +1$ defines the unity component and $\lambda_u = -1$ correspond to the inversion. ▶

73. Half a manifold and half not...

In conclusion, the geodesics trajectories of the manifold T^2 is a good and natural example of a diffeological space which is half manifold and half not. The projection $\mathrm{pr}_1 : \mathrm{Geod}(T^2) \to S^1$ looks like a fiber bundle but is just a bundle since the fibers are not diffeomorphic to each other. The group of diffeomorphisms of this space turns out to be the group of automorphisms of the projection, which in particulier shows the rigidity of the structure. The orbits of the group of diffeomorphisms mix ordinary circles — the orbits of rational geodesics — with irrational tori, which is an interesting situation.

The D-topology of the space of geodesics of T^2 is poor, since the largest Hausdorff quotient is the circle S^1 only; but the diffeology, again, is able to discriminate between the non equivalent parts. That gives an interesting Klein stratification where strata are not all manifolds, actually thet are almost never manifolds except a countably infinite subset of them.

Notes

The Diffeomorphisms of the Square

In this note we shall see how diffeology, understood as the geometry of the group of diffeomorphisms in the sense of Felix Klein, fulfills its duty concerning the full square $\mathrm{Sq} = [0,1]^2 \subset \mathbf{R}^2$.

According to Klein's program [**Kle72**], a geometry must be understood as the action of a group (called the *principal group*) on some space. As an example, Euclidean geometry is the action of the Euclidean group on a Euclidean space, affine geometry, the action of the affine group etc. The example of the closed square discussed here, in the Diffeology framework, seems to support the point of view that "differential geometry" is the geometry of the group of diffeomorphisms, in the sense of Klein.

277. Klein decomposition. Let $\mathrm{Sq} = [0,1]^2 \subset \mathbf{R}^2$ be equipped with the subset diffeology. The decomposition of the square under the group of diffeomorphisms gives the expected three orbits:

(1) The 4 corners: $\mathrm{NO} = (0,1)$, $\mathrm{NE} = (1,1)$, $\mathrm{SO} = (0,0)$ and $\mathrm{SE} = (1,0)$.
(2) The 4 vertices $\mathrm{B} =]0,1[\times\{0\}$, $\mathrm{T} =]0,1[\times\{1\}$, $\mathrm{L} = \{0\}\times]0,1[$ and $\mathrm{R} = \{1\}\times]0,1[$.
(3) The interior $\mathring{\mathrm{Sq}} =]0,1[^2$.

Note 1. The quotient diffeology on

$$\mathrm{Sq}/\mathrm{Diff}(\mathrm{Sq}) = \{\mathrm{Corners}, \mathrm{Vertices}, \mathrm{Interior}\}$$

is of course not the discrete diffeology, it captures the combinatorial structure of the orbits.

Note 2. that under homeomorphisms there are only 2 orbits, the border and the interior. In our case, the differential structure is obviously indispensable.

☙ Proof. (1) Let us show that, Sq is embedded in $\mathbf{R}^2$. That is, the D-topology of the induction $\mathrm{Sq} \subset \mathbf{R}^2$ coincides with the induced topology of $\mathbf{R}^2$. For any subset $\mathrm{U} \subset \mathrm{Sq}$ open for the induced topology, there exists an open $\mathcal{O} \subset \mathbf{R}^2$ such that $\mathrm{U} = \mathcal{O} \cap \mathrm{Sq}$. For all plots P in Sq, $\mathrm{P}^{-1}(\mathrm{U}) = \mathrm{P}^{-1}(\mathcal{O})$ is open, because plots are continuous. On the other hand, let $\mathrm{U} \subset \mathrm{Sq}$ be a D-open. $s^{-1}(\mathrm{U})$ is open, where $s\colon \mathbf{R}^2 \to \mathrm{K}^2$ is the map $s(x_1, x_2) = (x_1^2, x_2^2)$. $s^{-1}(\mathrm{U}) \upharpoonright \mathrm{Sq}$ is open for the induced topology of $\mathbf{R}^2$. Now, the map s restricted to Sq is an homeomorphism. Hence, since $\mathrm{U} = s(s^{-1}(\mathrm{U}) \upharpoonright \mathrm{Sq})$, U is open for the induced topology of $\mathbf{R}^2$. Therefore the D-topology of the induction coincides with the induced topology.

(2) Now, let us show that a diffeomorphism of Sq cannot send a point of the border into the interior. Let $f\colon \mathrm{Sq} \to \mathrm{Sq}$ be a diffeomorphism for the subset diffeology. Hence f is a homeomorphism for the D-topology. Let us assume that f maps $x \in \mathrm{L}$ to $f(x) \in \mathring{\mathrm{Sq}}$. Obviously f induces a homeomorphism $\tilde{f}\colon \mathrm{Sq} \setminus \{x\} \to \mathrm{Sq} \setminus \{f(x)\}$. Since $x \in \mathrm{L}$, we have that $\mathrm{Sq} \setminus \{x\}$ is convex, therefore homotopy equivalent to a point. On the other hand, we can construct a homotopy equivalence $\mathrm{Sq} \setminus \{f(x)\} \simeq \partial\mathrm{Sq} \simeq \mathrm{S}^1$. Thus, we get $\{\text{point}\} \simeq \mathrm{S}^1$, which is a contradiction.

(3) A diffeomorphism from the square must send corners into corners. Let f be a diffeomorphism on Sq, such that $f \upharpoonright \mathrm{L}$ takes values in $\mathrm{L} \cup \mathrm{SO} \cup \mathrm{B}$ and the restriction of f to L is defined as follows,

$$f(te_2) = \begin{cases} \phi(t)e_1, & \text{if } 0 < t < \frac{1}{2} \\ \mathrm{SO}, & \text{if } t = \frac{1}{2} \\ \psi(t)e_2, & \text{if } \frac{1}{2} < t < 1 \end{cases}$$

where e_1 and e_2 are the vectors of the canonical basis of $\mathbf{R}^2$.

Thus, for all $p > 0$,

$$\lim_{t\to\frac{1}{2}^-} f^{(p)}(te_2) = \phi^{(p)}(t)e_1 \quad \text{and} \quad \lim_{t\to\frac{1}{2}^+} f^{(p)}(te_2) = \psi^{(p)}(t)e_2.$$

Hence by continuity, for all $p > 0$, $\phi^{(p)}\left(\frac{1}{2}\right) = \psi^{(p)}\left(\frac{1}{2}\right)$.

Therefore, f is flat at $\frac{1}{2}$. The restriction of f^{-1} to $\mathrm{L} \cup \mathrm{SO}$ is a local diffeomorphism of half space with the subset diffeology. By [**TB**, § 4.14] (a consequence of [**Whi43**]), there exists $\mathrm{F} \in \mathcal{C}^\infty(]-\varepsilon, 1[, \mathbf{R})$ such that, $f^{-1} \restriction \mathrm{L} \cup \mathrm{SO} = \mathrm{F} \restriction \mathrm{L} \cup \mathrm{SO}$. We have $f^{-1} \circ f(t) = t$, for all $t > \frac{1}{2}$. Thus, we have $(f^{-1})'(f(t))f'(t) = 1$. But, $(f^{-1})'(f(t))f'(t) = \lim_{t\to\frac{1}{2}} \mathrm{F}'(f(t))f'(t) = 0$ which is a contradiction. Therefore 4 corners of Sq is an orbit of Diff(Sq). ▶

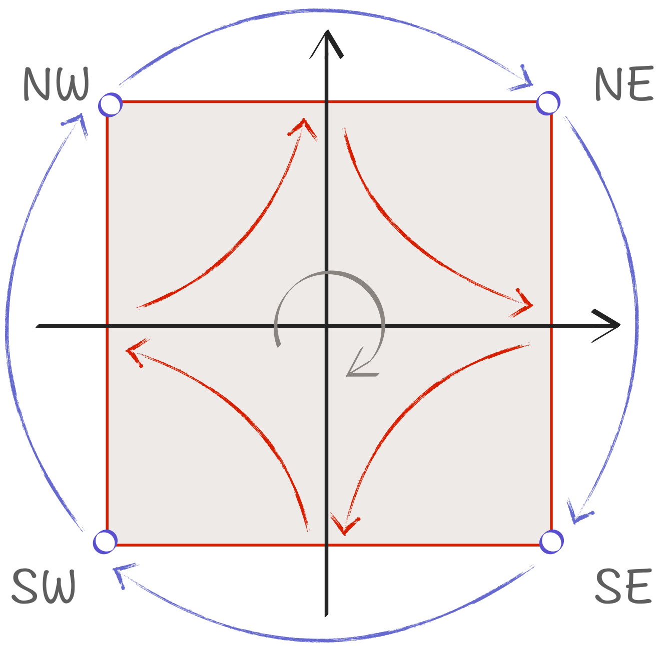


Figure 38. The diffeomorphisms of the square.

Diffeological Spaces are Locally Connected

Something that should have been said a long time ago: Every diffeological space is locally connected for the D-topology.

Let us recall that every diffeological space X naturally owns a topology called the D-topology. It was defined originally in [**Igl85**] but see also [**TB**, § 2.8]. It is the finest topology for which the plots are continuous. A subset $\mathcal{O} \subset X$ is open for the D-topology, or is a *D-open*, if $P^{-1}(\mathcal{O})$ is open in dom(P) for all plots P in X.

It has been shown that a the connected components for the D-topology are the path-connected components, and that the space X is the sum of its connected components [**TB**, § 5.7, 5.8].

We can say more about that, but first we recall one of the definitions of *locally connected spaces*.

278. Definition. *A topological space is said to be* locally connected *if every open subset is the sum of its components, for the induced topology.*

Then:

279. Proposition. *Every diffeological space is locally connected. That means that every D-open subset $\mathcal{O}$ is the sum of its connected components for the D-topology induced on $\mathcal{O}$. Since connected components for the D-topology coincide with pathwise connected components, every connected diffeological space X is locally pathwise connected.*

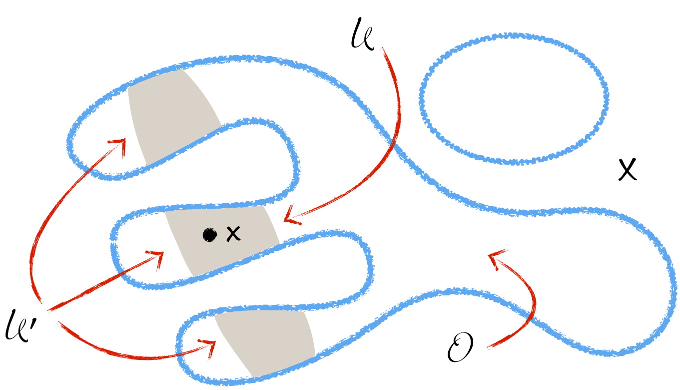
☞ Proof. Let us prove that D-open subsets $\mathcal{O} \subset X$ are embedded [**TB**, § 2.14]. That is, every D-open subset $\mathcal{U}$ of the subspace $\mathcal{O} \hookrightarrow X$ is the imprint of a D-open subset $\mathcal{U}'$ of X, *i.e.*

$$\mathcal{U} = \mathcal{U}' \cap \mathcal{O},$$

where $\mathcal{U}'$ is D-open in X and $\mathcal{U}$ is D-open in $\mathcal{O}$ for its D-topology as a subspace. So, on the one hand, since $\mathcal{U}$ is open for the D-topology of the subspace $\mathcal{O}$ of X, then for every plot P in $\mathcal{O}$, that is, for every plot P in X taking its values in $\mathcal{O}$, $P^{-1}(\mathcal{U})$ is open. On the other hand, let P be any plot in X. since $P^{-1}(\mathcal{U}) \subset P^{-1}(\mathcal{O})$, $P^{-1}(\mathcal{U}) = [P \restriction P^{-1}(\mathcal{O})]^{-1}(\mathcal{U})$. But, $P^{-1}(\mathcal{O})$ is open since $\mathcal{O}$ is D-open, then $P \restriction P^{-1}(\mathcal{O})$ is still a plot in X but taking its values in $\mathcal{O}$. That is, a plot in $\mathcal{O}$. Thus, $[P \restriction P^{-1}(\mathcal{O})]^{-1}(\mathcal{U})$ is open, and so is $P^{-1}(\mathcal{U})$. Thus, for all plots P in X, $P^{-1}(\mathcal{U})$ is open. Therefore, $\mathcal{U}$ is D-open in X, and since $\mathcal{U} = \mathcal{U} \cap \mathcal{O}$, $\mathcal{U}$ as D-open $\mathcal{O}$ is the imprint of $\mathcal{U}$ on $\mathcal{O}$ as D-open in X. In conclusion, every D-open subset of X is embedded.

In summary, the intersection above $\mathcal{U} = \mathcal{U}' \cap \mathcal{O}$ read as follows: every D-open subset in $\mathcal{O}$ is a D-open in X, since it is the intersection of two D-open subset of X.

Now, as a subspace of X, $\mathcal{O}$ is the sum of its component which are D-open subset in $\mathcal{O}$, and then, D-open subset of X. Thus, $\mathcal{O}$ is the sum of its components for the D-topology of X. Therefore, X is locally connected for the D-topology. ►



Vague Adjunction of a Point to a Space

We add a point to a diffeological space, counting for nothing, and we look at the consequences.

Consider a diffeological space X and $\mathcal{D}$ be its diffeology. Let

$$\bar{X} = X \cup \{\omega\},$$

where ω is an arbitray point not contained in X. We define on $\bar{X}$ the following set $\mathcal{D}'$ of parametrizations:

280. Definition. A parametrization $P : U \to \bar{X}$ belongs to $\mathcal{D}'$ if for all r_0 such that $P(r_0) \in X$, there exists an open neighborhood V of r_0 such that $P \upharpoonright V \in \mathcal{D}$. No condition is required for $P(r_0) = \omega$.

281. **Proposition 1** A parametrization $P : U \to \bar{X}$ belongs to $\mathcal{D}'$ if and only if its restriction on $P^{-1}(X)$ is a plot in X. This means in particular that $P^{-1}(X) \subset U$ is an open subset. As a corollary, $\mathcal{D}'$ is a diffeology on $\bar{X}$.

Proof. If $P \upharpoonright P^{-1}(X)$ is a plot in X then it satisfies the condition above and belongs then to D'. Conversely, let $P \in \mathcal{D}'$ and $\mathcal{O} = P^{-1}(X)$: since $P \in \mathcal{D}'$, for all $r \in \mathcal{O}$ there exists a neighborhood V of r such that $P \upharpoonright V$ belongs to $\mathcal{D}$. This implies that $P(V) \subset X$ and then $V \subset \mathcal{O}$. Thus, $\mathcal{O}$ is a union of open subsets of an Euclidean domain, it is then open and $P \upharpoonright P^{-1}(X)$ is a plot in X. ▶

282. **Proposition 2.** $\bar{X}$ is connected. Precisely, let $P : \mathcal{B} \to \bar{X}$ be the parametrization defined by:

- $\mathcal{B}$ is an open ball.
- $P(0) = \omega$.
- $P(\mathcal{B} - \{0\}) = x$, where $x \in X$ is any point.

Then, P is a plot connecting any point x in X to ω. Think of $\mathcal{B} = \mathbf{R}$.

Proof. Indeed $\mathcal{B} - \{0\}$ is open, and $P \upharpoonright \mathcal{B} - \{0\}$ is constant, then $P \upharpoonright \mathcal{B} - \{0\}$ is a plot, and since it takes its value in X, P is a plot in $\bar{X}$. Now, since $\mathcal{B}$ is connected, x and ω are in the same connected component, for all $x \in X$. ▶

283. Proposition 3. The point ω is closed in $\bar{X}$.

Proof. The set $\{\omega\}$ is closed if and only if $X \subset \bar{X}$ is D-open. The set $X \subset \bar{X}$ is D-open if and only if for all plots P in $\bar{X}$, $P^{-1}(X)$ is open. But this is exactly what we have seen above, in the proof of Proposition 1. Therefore, $X \subset \bar{X}$ is D-open and $\{\omega\} \in \bar{X}$ is D-closed. ▶

284. Proposition 4. Every point x in X is in the neighborhood of ω. In other words, ω has only one neighborhood, the space $\bar{X}$ itself.

Proof. Let Ω be an open neighborhood of ω. That is, for every plot P in $\bar{X}$, $P^{-1}(\Omega)$ is open. Let $x \in X$ and let $P : \mathcal{B} \to \bar{X}$ be a special plot defined in the second article, that sends 0 to ω and the rest of the ball on x. Since $P^{-1}(\Omega)$ is open and 0 is sent to ω, there exists a small open ball $\mathcal{B}'$ centered at 0 such that its image by P is contained in Ω. By construction $P(\mathcal{B}' - \{0\}) = x \in \Omega$. Therefore Ω contains every point in X. That is, $\Omega = \bar{X}$. ▶

285. Remark. This kind of situation is not unique to diffeology. Consider for example the quotient of $\mathbf{R}^n$ by the action of $]0, \infty[$, $(\alpha, x) \mapsto \alpha x$, where $(\alpha, x) \in]0, \infty[\times \mathbf{R}^n$. For the quotient topology of $\mathbf{R}^n/]0, \infty[$, any neighborhood of class(0) contains the whole quotient. Set theorethically, the quotient is the union of $S^{n-1} = (\mathbf{R}^n - \{0\})/]0, \infty[$ with class(0).

Embedding a Diffeological Space Into its Powerset

In this note we shall see that the natural inclusion of a diffeological space into its powerset is an embedding. And a closed embedding if the space is Hausdorff.

The main ingredient is the *Powerset diffeology* of a diffeological space, defined in the Exercise 62 of the textbook [TB]. We recall, in the first article, the results of this exercise.

286. The inclusion map. Let X be a diffeological space and $\mathfrak{P}(X)$ be its powerset, equipped with the powerset diffeology. The natural *inclusion map*

$$j\colon X \to \mathfrak{P}(X) \quad \text{with} \quad j\colon x \mapsto \{x\},$$

is smooth and is an induction [TB, § 1.29].

☞ Proof. Let $P\colon U \to X$ be a plot in X, we have to check that $j \circ P\colon r \mapsto j(P(r)) = \{P(r)\}$ is a plot for the powerset diffeology.

For all $r_0 \in U$ and for all plots Q_0 with value in $\{P(r_0)\}$, which is necessarily constant, we define the following family of constant plots:

$$r \mapsto Q_r : \mathrm{dom}(Q_0) \to X, \text{ with } Q_r(s) = P(r).$$

Since $\mathrm{val}(Q_r) \subset j(P(r)) = \{P(r)\}$, and $(r, s) \mapsto Q_r(s) = P(r)$ is clearly smooth, the conditions to be a powerset plot are satisfied by $j \circ P$. Thus, j is smooth.

Let us check that j is now an induction. Let $\mathrm{P}\colon \mathrm{U} \to \mathfrak{P}(\mathrm{X})$ be a powerset plot with value in $j(\mathrm{X})$, we have to check that $j^{-1} \circ \mathrm{P}$ is a plot in X. Indeed, for all $r \in \mathrm{U}$ there exists $x_r \in \mathrm{X}$ such that $\mathrm{P}(r) = \{x_r\}$. For any fixed $r_0 \in \mathrm{U}$ and for any plot Q_0 such that $\mathrm{val}(\mathrm{Q}_0) \subset \mathrm{P}(r_0) = \{x_{r_0}\}$, there exists an open neighborhood V of r_0 and a smooth family $r \mapsto \mathrm{Q}_r$ of plots, defined on V, such that $\mathrm{val}(\mathrm{Q}_r) \subset \mathrm{P}(r) = \{x_r\}$. Thus, $(s, r) \mapsto \mathrm{Q}_r(s) = x_r = j^{-1} \circ \mathrm{P}$ is locally smooth, and $r \mapsto x_r$ is a plot of X. ▶

287. Embedding X in its powerset. The inclusion $j\colon \mathrm{X} \to \mathfrak{P}(\mathrm{X})$ is not just an induction, it is an embedding [TB, § 2.13]. That is, the pullback of the D-topology of $\mathfrak{P}(\mathrm{X})$ on X coincides with its D-topology.

☞ Proof. We have to prove that, for any D-open subset $\mathcal{O} \subset \mathrm{X}$, there exists an D-open set $\mathcal{O}'$ in $\mathfrak{P}(\mathrm{X})$ such that $j(\mathcal{O}) = j(\mathrm{X}) \cap \mathcal{O}'$.

Let us define

$$\mathcal{O}' = \{\mathrm{A} \in \mathfrak{P}(\mathrm{X}) \mid \mathrm{A} \cap \mathcal{O} \neq \varnothing\}.$$

Clearly $j(\mathcal{O}) = j(\mathrm{X}) \cap \mathcal{O}'$. It remains to prove that $\mathcal{O}'$ is D-open in $\mathfrak{P}(\mathrm{X})$. Then, for any plot $\mathrm{P}\colon \mathrm{U} \to \mathfrak{P}(\mathrm{X})$, we have to prove that

$$\mathrm{P}^{-1}(\mathcal{O}') = \{r \in \mathrm{U} \mid \mathrm{P}(r) \cap \mathcal{O} \neq \varnothing\}$$

is open. Let $r_0 \in \mathrm{P}^{-1}(\mathcal{O}')$, thus $\mathrm{P}(r_0) \cap \mathcal{O} \neq \varnothing$. Pick $x_0 \in \mathrm{P}(r_0) \cap \mathcal{O}$ and the constant plot $\mathrm{Q}_0(s) = x_0$, there exists a smooth family of plots Q_r for r near r_0 such that $\mathrm{val}(\mathrm{Q}_r) \subset \mathrm{P}(r)$ and $(s, r) \mapsto \mathrm{Q}_r(s)$ is smooth. Since $\mathcal{O}$ is open, by continuity of $\mathrm{Q}_r(s)$, it exists a product of two balls $\Omega = \mathrm{B}(s, \varepsilon) \times \mathrm{B}(r_0, \eta)$ (s is arbitrary choosen) such that for all $(s, r) \in \Omega$, $\mathrm{Q}_r(s) \in \mathcal{O}$, the condition $\mathrm{val}(\mathrm{Q}_r) \subset \mathrm{P}(r)$ implies that $\mathrm{P}(r) \cap \mathcal{O} \neq \varnothing$. Thus, $\mathrm{B}(r_0, \eta) \subset \mathrm{P}^{-1}(\mathcal{O}')$, that is, $\mathrm{P}^{-1}(\mathcal{O}')$ is open, and therefore, $\mathcal{O}'$ is D-open. ▶

288. The case of the empty set. Let $\mathfrak{P}(\mathrm{X})^\star$ be the subset of non empty sets in X,

$$\mathfrak{P}(\mathrm{X})^\star = \mathfrak{P}(\mathrm{X}) - \big\{\{\varnothing\}\big\}.$$

Then, equipped with the powerset diffeology, $\mathfrak{P}(\mathrm{X})$ is the vague adjunction of the singleton $\{\varnothing\}$ to $\mathfrak{P}(\mathrm{X})^\star$, as defined in the previous note in “Vague adjunction of a point to a space”. That implies in particular that, for the D-topology, $\{\varnothing\} \in \mathfrak{P}(X)$ is closed, and $\mathfrak{P}(X)$ is the only neighborhood of $\{\varnothing\}$.

Proof. Indeed, for a parametrization P in $\mathfrak{P}(X)$ and a point $r_0 \in \mathrm{dom}(P)$, if $P(r_0) = \{\varnothing\}$ then the condition of the Powerset Diffeology is empty. ▶

289. X in $\mathfrak{P}(X)$. If X is Hausdorff for the D-topology, then its image J(X) in $\mathfrak{P}(X)^\star$ is closed. Or, $J(X) \cup \{\varnothing\}$ is closed in $\mathfrak{P}(X)$.

Proof. Let us show that $\mathfrak{P}(X)^\star - j(X)$ is open,

$$\mathfrak{P}(X)^\star - j(X) = \left\{ A \subset X \mid \exists \{x, y\} \subset A, x \neq y \right\} .$$

Consider a plot $P : U \to \mathfrak{P}(X)^\star$, and $r_0 \in P^{-1}(\mathfrak{P}(X)^\star - j(X))$. Suppose now $\{x_0, y_0\} \subset P(r_0)$ with $x_0 \neq y_0$. Let Q_0 and Q'_0 be two constant plots such that : $\mathrm{val}(Q_0) = \{x_0\}$ and $\mathrm{val}(Q'_0) = \{y_0\}$, so there exists two smooth family $U \supset V \ni r \mapsto Q_r$ and $U \supset V' \ni r \mapsto Q'_r$ such that $\mathrm{val}(Q_r) \subset P(r)$ and $\mathrm{val}(Q'_r) \subset P(r)$. As X is supposed to be Hausdorff, we may choose $\mathcal{O}$ and $\mathcal{O}'$ two disjoint D-open neighborhoods of respectively x_0 and y_0. Given s_0 in a common domain of Q_r and Q'_r, the intersection of the pre-images of $\mathcal{O}$ and $\mathcal{O}'$ by the continuous maps $(r, s) \mapsto Q_r(s) \in X$ and $(r, s) \mapsto Q'_r(s) \in X$ defines an open neigborhood $W \times S$ of (r_0, s_0), with $W \subset V \cap V'$, such that $Q_r(s) \in \mathcal{O}$ and $Q'_r(s) \in \mathcal{O}'$ for any $(r, s) \in W \times S$. As $\mathrm{val}(Q_r) \subset P(r)$ and $\mathrm{val}(Q'_r) \subset P(r)$, the condition $\mathcal{O} \cap \mathcal{O}' = \varnothing$ insures that $P(r)$ includes at least two distinct points of X for all $r \in W$. Hence $W \subset P^{-1}(\mathfrak{P}(X)^\star - j(X))$. Thus, $P^{-1}(\mathfrak{P}(X)^\star - j(X))$ is open, as a union of open subsets. Therefore, $\mathfrak{P}(X)^\star - j(X)$ is D-open and $j(X)$ is closed in $\mathfrak{P}(X)^\star$. ▶

Foliations and Diffeology

Spaces of leaves of foliations on manifolds are the first examples we can think about when it comes to diffeology. Paradoxically, with the exception of Kronecker's foliation of a torus by an irrational line, I haven't delved into this class of examples in the textbook. In [**IL90**] we treated the particular case of the foliation of a n-dimensional torus T^n by an irrational hyperplane. In this note we go a little bit further about what diffeology can say about the space of leaves of a foliated manifold?

Let us begin by recalling what a foliation is [**Law74**]. Let M be an n-manifold [**TB**, 4.1]. We say that a partition $\mathcal{L}$ of M, into connected subspaces, is a *foliation* if there exists an atlas of charts

$$\phi : \mathbf{R}^k \times \mathbf{R}^m \supset U \to M, \text{ with } k = n - m,$$

such that, for all $L \in \mathcal{L}$, the preimage by ϕ of each connected component of $L \cap \phi(U)$ is a vertical slice $U_r = U \cap [\{r\} \times \mathbf{R}^m]$, for some $r \in \mathbf{R}^k$, see [**TB**, 1.33, 5.7]. In simple words, a foliation is a partition that looks like locally to a product of Euclidean spaces.

The elements L of $\mathcal{L}$ are called the *leaves* of the foliation. The charts ϕ above are called *adapted* to the foliation $\mathcal{L}$. Note that the leaves, equipped with the subset diffeology, are m-dimensional submanifolds [**TB**, 4.4]. Indeed, with the notations above, the restrictions $\phi \upharpoonright U_r$ are local diffeomorphisms, defined on U_r, from $\mathbf{R}^m$ to L.

290. Transversals. We call a *tranversal* of $\mathcal{F}$ any k-plot $Q : V \to M$ such that there exists an adapted chart $\phi : V \to M$ for which $Q(r) = \phi(r, 0)$. A plot P in M is said to be *transversal* if it takes its values in the values of a transversal Q.

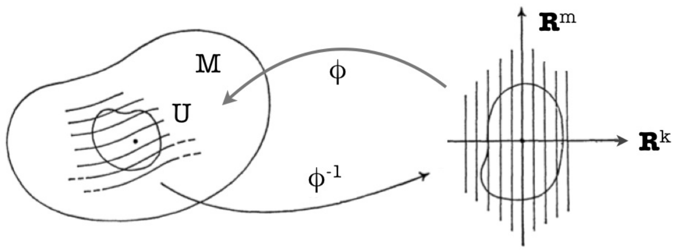


Figure 39. Rectification of the flow.

Now, let $\mathcal{L}$ be equipped with the quotient diffeology [TB, 1.50], let $\pi : M \to \mathcal{L}$ be the projection, that is, $\pi(x) = L$ if and only if $x \in L \in \mathcal{L}$.

291. The space of leaves. Let M be a second countable Hausdorff manifold, and $\mathcal{L}$ be a foliation on M.

(1) Let $Q : V \to M$ be a transversal, then for all $L \in \mathcal{L}$, $L \cap Q(V)$ — or $\{r \in V \mid Q(r) \in L\}$ — is at most countable, thus, either empty or (diffeologically) discrete.
(2) There are generating families of $\mathcal{L}$, equipped with the quotient diffeology, made of parametrizations $\mathcal{Q} = \pi \circ Q$, where Q is a transversal plot.
(3) The dimension [TB, 2.22] of the quotient $\mathcal{L}$ is constant and equal to $k = n - m$.

☞ Proof. Let us deal the first statement. Since Q is the restriction of a chart ϕ, it is equivalent to consider $L \cap Q(V)$ or the set $\{r \in V \mid Q(r) \in L\}$, they are isomorphic. Let us assume that $\{r \in V \mid Q(r) \in L\}$ is not discrete, that is, there exists a 1-plot $t \mapsto r(t)$ in V, defined on some interval J, such that $Q(r(t)) \in L$ for all t, and there are $a, b \in J$ with $a \neq b$ and $Q(a) \neq Q(b)$. ▶

Klein Stratification of Diffeological Spaces

In this note we see that every diffeological space is naturally stratified by the action of its diffeomorphisms.

74. Klein stratifications

As we already know, diffeology is a flexible category, stable by any set-theoretic operation. In particular diffeology gives a simple and natural access to singularities, as many examples already have shown [DI83, IKZ10, PIZ15], even when the space is topologically trivial the diffeology can be relevant, which is the case for irrational tori for example [DI83].[1]

In the case of irrational torus, the singularity lies in the global nature of the space, and comes from the discrepancy between its trivial topology and its non-trivial diffeology. On the other hand, from a local point of view, the diffeology of a space encodes in a strong way, the internal singularities of the space. They are revealed by the action of the diffeomorphisms. For example, in a square a diffeomorphism can exchange only vertices and edges and fixes the interior.

[1]This remarkable property has important consequences since it allows theorems that could not exist otherwise. For example, one proved that every symplectic manifold is an orbit of the linear coadjoint action of a central extension of the group of hamiltonian diffeomorphisms by the torus of periods of the symplectic form, which is in general not a manifold but trivial as topological space [DIZ22].

This remark gave rise to the first definition of Klein strata of a diffeological space, as the orbits of the group of diffeomorphisms. We recall this definition from [**TB**, § 1.42].

292. **Definition 1.** Let X be a diffeological space, the *Klein strata* of X are defined as the orbits of Diff(X), its group of diffeomorphisms.

An important remark that did not appear in the book in the section 1.42 devoted to Klein's strata is the following:

293. **Proposition 1.** The Klein strata of a diffeological space X, defined as the orbits of its group of diffeomorphisms, form a stratification in the sense that the closure of a Klein stratum, for the D-topology, is a union of Klein strata. In other words, if $\mathcal{O}$ is an orbit of Diff(X), then there exists a subset $\Sigma \subset X$ such that

$$\overline{\mathcal{O}} = \cup_{x \in \Sigma} \mathcal{O}_x,$$

where $\mathcal{O}_x$ denotes the orbit of x and $\overline{\mathcal{O}}$ the closure of $\mathcal{O}$.

The D-topology has been defined originally in [**Igl85**] and also in [**TB**, § 2.8]. This is the finest topology for which the plots are continuous. A subset $\mathcal{O} \subset X$ is open for the D-topology, or is a *D-open*, if $P^{-1}(\mathcal{O})$ is open in dom(P) for all plots P in X. The previous proposition can be stated as follows:

294. **Proposition 1 bis.** Let $\mathcal{O}_x$ and $\mathcal{O}_y$ be two orbits by the group Diff(X). If $x \in \overline{\mathcal{O}_y}$ and $x' \in \mathcal{O}_x$, then $x' \in \overline{\mathcal{O}_y}$.

Proof. The proof is straightforward: since diffeomorphisms are homeomorphisms [**TB**, § 2.9], the closure relation is preserved by diffeomorphisms. The following is just given to make this statement obvious.

Let U' be a D-open neighborood of x'. Since $x' = f(x)$, with $f \in \mathrm{Diff}(X)$, $U = f^{-1}(U')$ is a D-open neighborhood of x. Then, there exists $z = g(y) \in \mathcal{O}_y \cap U$. Thus $z' = f(z)$ belong to $U' = f(U)$ and $z' \in \mathcal{O}_y$, since $z \in \mathcal{O}_y$ and $z' = f(z)$. Therefore $U' \cap \mathcal{O}_y \neq \emptyset$. ▶

Consider the example of the square Sq described in Figure 38. The group Diff(Sq) has three orbits:

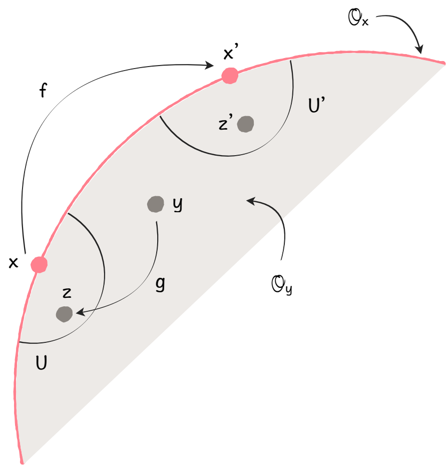


Figure 40. The frontier condition.

1. the 4-corners-orbit;
2. the 4-edges-orbit;
3. the interior-orbit.

The first remark we can do here is that the corners orbits, and the edges orbit, are not connected but their elements can be treated separately. Indeed, it is sufficient to redefine the Klein strata:

295. Definition 1 bis. Let X be a diffeological space, the connected *Klein strata* of X are defined as the orbits of $\mathrm{Diff}(X)^0$, the identity component of the group of diffeomorphisms of X.

Connected Klein strata are indeed connected. They are the images of a connected diffeological group by a smooth *orbit applications* $\hat{x} : f \mapsto f(x)$.

In the case of the square there are now nine strata:

1. Four corners: NE, SE, SW, NW;
2. Four edges: NE—SE, SE—SW, SW—NW, NW—NE;
3. One interior.

As a remark, the partition of X into connected Klein strata is still a stratification in the sense that it satisfies the frontier condition, for the same reason of proposition 1.

296. Proposition 2. The connected Klein strata of a diffeological space X, defined as the orbits of the identity component of its group of diffeomorphisms, form a stratification in the sense that the closure of a Klein stratum is a union of Klein strata.

These few previous considerations suggest a broadening of the definition of these stratifications:

297. Definition 2. Let X be a diffeological space, Let G be any diffeological group acting on X by a smooth homomorphism $g \mapsto g_X$. The orbits of the action of G on X form a *geometric stratification* that satisfies the frontier condition.

Proof. Identical to Proposition 1. ▶

Now, the notion of stratification goes hand in hand with that of singularity. The idea of singularity is by definition local. We consider then the local geometry of a diffeological space: it is defined at each point by the germ of the diffeology there.[2] The local geometry at each point is preserved by the action of local diffeomorphisms, which is no more a group but a so-called pseudo-group, we denote it by $\mathrm{Diff}_{\mathrm{loc}}(X)$. Local diffeomorphisms can exchange only points with the same local geometry. That leads to the following definition:

298. Definition 3. Let X be a diffeological space. We call *local Klein strata* the orbits of its local diffeomorphisms.

In other words, Klein strata gather the points that share the same local geometry.

299. Proposition 3. The local Klein strata of a diffeological space X, defined as the orbits of the local diffeomorphisms, form a stratification in the sense that the closure of a Klein stratum is a union of Klein strata.

[2]That can be defined precisely.

☞ Proof. Identical to Proposition 1 since local diffeomorphisms are defined on D-open subsets. ▶

75. Singularities

Local Klein strata are associated with the idea of *singularity*. In some sense they capture the singular points of the diffeological space. However, the singularity here is a relative concept, a point is not singular by itself but relatively to others. This means, in the example of the square, that the corners are singular to the interior points, as are the edges, but they are not equivalent to each other.

The precise definition of singularity dwells in the definition of preorder associated with every geometric stratification, and eventually with the local Klein strata.

300. Proposition 4. The binary relation defined on the space of local Klein strata by

$$\mathcal{O} \preceq \mathcal{O}' \quad \text{iff} \quad \mathcal{O} \subset \overline{\mathcal{O}'}$$

is a natural *preorder*, that is, reflexive and transitive. One can say that $\mathcal{O}$ is *singular* with respect to $\mathcal{O}'$ if $\mathcal{O} \preceq \mathcal{O}'$. The set of strata is called a PrOSet.

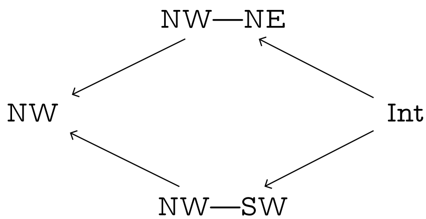
It appears that the space of local Klein strata, equipped with the quotient diffeology, is an Alexandrov topological space for this preorder. This preorder is a partial order (reflexive, transitive and antisymetric) if and only if the D-topology of the space of strata is T_0-separated. It has been shown that this is equivalent for the strata to be locally closed when the projection map is open, see [**SY19**] for example. In this case we say that the space of strata is a POSet.

301. Example of a PrOSet. The *solenoid* action of $\mathbf{R}$ on the 2-torus:

$$\underline{t}(z, z') = \left(ze^{2i\pi t}, z'e^{2i\pi\alpha t}\right)$$

gives a POset if $\alpha \in \mathbf{Q}$ and only a PrOSet otherwise, if $\alpha \in \mathbf{R} - \mathbf{Q}$. In this case, every stratum is in the closure of every other one, that is, the whole torus T^2.

302. Example of a POSet The space of strata of the square is a POset. There are four minimal strata of the following type:



where the arrow is for $\preceq$. The vertices NW, NE, SE, SW are the minimal points.

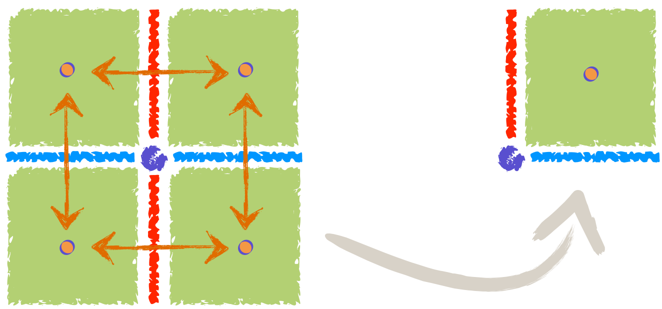


Figure 41. The corner orbifold $\mathbf{R}^2/\{\pm 1\}^2$.

Note. An interesting application of this subject would be the analysis of the Klein stratification of the space of orbits of a manifold M by the action of a compact group G. In particular, to compare the stratification of M by type of stabilizer and the Klein stratification of M/G. We have studied, with Serap Gürer, the Klein stratification of an orbifold [GIZ22], see Figure 41 for example, and we have proved that it coincides with the orbit type stratification.

Lagrange's Equations of Motion

In this note[1] I apply the Lagrange method of variation of constant to Newton's equations, by considering the force involved in the motion of a point as a perturbation of the absence of forces.

This note should be seen as a *Récréation*, a break from abstract diffeology to return to the physics for which diffeology was specifically developed. It is not that diffeology is used in this note, which is a reflection on Lagrange's equations of motion, in relation to what is usually called the "Hamiltonian formalism", but I hope that one day, in the near future, I'll have some examples where this is the case.

In a paper on Lagrange's work I explained how Lagrange introduced the first elements of symplectic geometry, when he applied his *method of variation of the constants* to the motion of the earth around the sun, see [**Lag08, Lag09, Lag10**]. In these memoirs, Lagrange regarded the motion of the earth as a curve in the space of its Keplerian elements, the points of the curve representing, at each time, the Keplerian motion that the earth would follow if the forces exerted by the other planets ceased at this instant. The motion of the planet is not anymore described then in space-time but in the *space of Keplerian motions*, that is, by the curve of its *tangential Keplerian motions* at each instant. The question then consists in expressing the differential equation satisfied by this curve and eventually in extracting

[1] Work in progress.

some informations on the stability of the system. It is when he established the nature of this curve that Lagrange introduced his *system of parentheses* which constitute the coordinates of the — today canonical — *symplectic structure* on the space of Keplerian motions. It was the birth of symplectic calculus and then symplectic geometry, see [PIZ98, PIZ02].

But the Lagrange's method of variation of constants applies in the first place to the basic Newton's equations where the force applied on a point is measured by the the deflexion of the tangential *inertial motion* (see Figure 42). The motion of the point is then regarded as a curve in the space of its tangent inertial motions of which we shall establish the differential equation. That is the original Lagrange interpretation of Newton's equations, this idea fully contained, but implicitely, in his second paper on the question in 1809, *op.cit.* I shall give in the following a modern construction of these *Lagrange equations of motion*.

Note. A physicist would say that the construction below is just a change of variables.[2] It is maybe the way that appears at first glance, but on a conceptual level, it does not just reduces to that. Newton equations are by construction Aristotelian, they do not assume only an absolute time — which is acceptable at this epoch — but also an absolute space, since Newton's equations are all about the second variation of the position in space. Meanwhile, since Galilee, we know that there is no such space, or if you prefer, there is no mechanical way, in the world we live in, to distinguish a rectilinear uniform motion from rest.[3] Then, Newton's equation, as they are taught in school, are even not compatible with Galilean principles,[4] but they are still not too far. We shall see how the Lagrange point of view transforms the Aristotelian Newton's equations into a system respectful of Galilean relativity.

[2]That is what happened to me during a talk at IHES.

[3]As a first approximation, of course, the reality being more complex.

[4]This is what generates so much confusion in school textbooks about Newton's equations.

76. The space of inertial motions

First of all, let us build the space of inertial motions on wich we will draw the curve representing the motion of our system. Considering the simple case of a free particle in $\mathbf{R}^3$, an inertial motion is just a uniform rectilinear motion, that is, a curve

$$\mu = [t \mapsto x] \quad \text{such that} \quad \frac{d^2x}{dt^2} = 0.$$

Let us denote by $\mathcal{M}_0$ this space of uniform rectilinear motions, $\mathcal{M}_0$ is a smooth manifold and we have a particular global chart:

$$\Phi : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathcal{M}_0, \quad \Phi(x, v) = [t \mapsto x + tv].$$

Now, let us consider the motion of a particle with mass m submitted to a force F, that is, a curve $t \mapsto x(t)$ in $\mathbf{R}^3$ satisfying the Newton equation:

$$m\frac{d^2x(t)}{dt^2} = \mathrm{F}(x(t), t).$$

The force F is assumed to be a smooth function defined on (an open subset of) $\mathbf{R}^3 \times \mathbf{R}$. At each intant t we can define the *tangent inertial motion* to the motion $[t \mapsto x(t)]$ as the uniform rectilinear motion $\mu(t)$ passing through $x(t)$ at the time t, with speed $v(t) = dx(t)/dt$, that is,

$$\mu(t) = [s \mapsto x(t) - tv(t) + sv(t)].$$

In other words, <u>the tangent inertial motion at the instant t is the inertial motion that would have the particle if the force vanished at this instant</u>.

Hence, the curve $t \mapsto \mu(t)$ represents the motion of the particle, but drawn in the space of inertial motions. Let us now determine the differential equation that this curve satisfies. We shall follow Lagrange hypothesis that there exists a *potential* Ω for the force, that is,

$$\mathrm{F}(x, t) = -\frac{\partial\Omega(x, t)}{\partial x}.$$

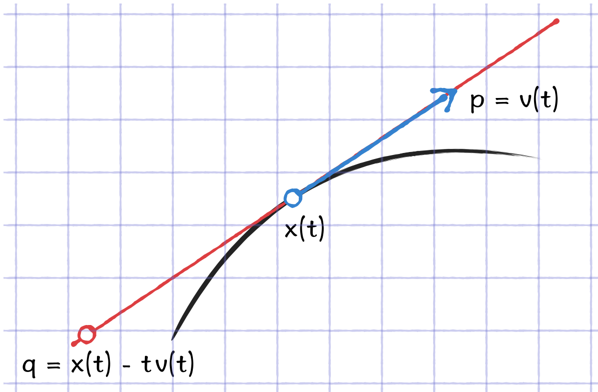


Figure 42. Tangent inertial motion.

In the chart Φ, the curve $t \mapsto \mu(t)$ writes:

$$\mu(t) = \Phi(q(t), p(t)) \quad \text{with} \quad \begin{cases} q(t) &= x(t) - tv(t), \\ p(t) &= v(t). \end{cases}$$

That gives the differential system

$$\frac{dq(t)}{dt} = \frac{dx(t)}{dt} - v(t) - t\frac{dv(t)}{dt} = -t\frac{dv(t)}{dt}$$
$$\frac{dp(t)}{dt} = \frac{dv(t)}{dt}.$$

Now, since the map $(x, v, t) \mapsto (q = x - tv, p = v, t)$ is a diffeomorphism, the potential Ω writes indifferently as a function of (x, v, t) or (q, p, t). From the relations between partial derivatives:

$$\frac{\partial\Omega}{\partial q} = \frac{\partial\Omega}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial\Omega}{\partial v}\frac{\partial v}{\partial q} + \frac{\partial\Omega}{\partial t}\frac{\partial t}{\partial q},$$
$$\frac{\partial\Omega}{\partial p} = \frac{\partial\Omega}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial\Omega}{\partial v}\frac{\partial v}{\partial p} + \frac{\partial\Omega}{\partial t}\frac{\partial t}{\partial p},$$

with $x = q + tp$, $v = p$, and noticing that, by hypothesis, Ω does not depend on v, we get eventually,

$$\frac{\partial\Omega}{\partial q} = \frac{\partial\Omega}{\partial x} \quad \text{and} \quad \frac{\partial\Omega}{\partial p} = t\frac{\partial\Omega}{\partial x},$$

which gives

$$\begin{aligned} \frac{dq(t)}{dt} &= -t\frac{dv(t)}{dt} &= -\frac{t}{m}\mathrm{F}(x(t),t) &= +\frac{t}{m}\frac{\partial\Omega}{\partial x} &= +\frac{1}{m}\frac{\partial\Omega}{\partial p} \\ \frac{dp(t)}{dt} &= \frac{dv(t)}{dt} &= +\frac{1}{m}\mathrm{F}(x(t),t) &= -\frac{1}{m}\frac{\partial\Omega}{\partial x} &= -\frac{1}{m}\frac{\partial\Omega}{\partial q}. \end{aligned}$$

Now, if we consider the symplectic structure, defined on $\mathcal{M}_0$, in the chart Φ, by

$$\omega(\delta\mu,\delta'\mu) = m\left[\langle\delta v,\delta' x\rangle - \langle\delta' v,\delta x\rangle\right] \quad \text{with} \quad \mu = \Phi(x,v), \qquad (\clubsuit)$$

we recognize in the differential system above the symplectic gradient of the potential Ω. And Newton equations write then:

$$\frac{d\mu(t)}{dt} = \mathrm{grad}(\Omega). \qquad \text{(L)}$$

We recall that the symplectic gradient of a real function f is defined by the identity

$$\mathrm{grad}(f) = -\omega^{-1}(df) \quad \text{or} \quad \omega(\mathrm{grad}\, f,\delta\mu) = -df(\delta\mu),$$

for all $\delta\mu \in \mathrm{T}_\mu\mathcal{M}_0$. The equations (L) are the *Lagrange equations of motion*. They are exactly described in the Lagrange second memoir of 1809, on the method of the variation of the constants in all the problems of Mechanics [**Lag09**]; only the notations change. Their construction leads to a few remarks:

<u>Note 1.</u> Lagrange's equations of motion respect Galilean relativity: the Galilean group acts naturally (by construction) on the space of inertial motions $\mathcal{M}_0$, that is, every element g in the Galilean group transform an inertial motion μ into another inertial motion $g_*(\mu) = g\circ\mu$. Moreover, the Galilean group preserves the symplectic structure on $\mathcal{M}_0$, $g^*(\omega) = \omega$, and the nature of the equations (L) are preserved under a Galilean transformation, precisely, for a curve $t \mapsto \mu(t)$, satisfying equations (L), the curve $g_*(\mu)$ satisfies

$$\frac{d[g_*(\mu)](t)}{dt} = \mathrm{grad}(g_*(\Omega)),$$

where $g_*(\Omega) = \Omega\circ g^{-1}$. So, Lagrange's version of Newton equations behaves correctly with respect to Galilean relativity.

Note 2. The term grad(Ω) in the equations above represents literally the force exerted on the point in this covariant Lagrange's symplectic framework. It is a well define geometrical object and the Lagrange equations of motion state the obvious, since Galilee: the force exerted on a particle is measured by the variation of the inertial motion. If the force vanished then $\mu = \mathrm{cst}$ *i.e.* in absence of force the particle follows an inertial motion.

Note 3. Lagrange's equations of motion look like Hamilton's equations, but they are not exactly the same. Indeed, Hamilton's equations involve the whole Hamiltonian $\mathrm{H} = p^2/2m + \Omega$ and not only the potential Ω, as it is the case in Lagrange's equations. In the Hamiltonian framework, the space involved is not regarded as the space of inertial motions (which is by construction Galilean respectful) but it is regarded as the space of initial conditions at some instant t_0, a *phase space*, (which is not Galilean respectful for the reason evoked above). This should also be seen in the light of what wrote Souriau in the introduction of his book "Structure des Systèmes Dynamiques" [Sou70]:

> "La mécanique analytique n'est pas une théorie périmée; mais il apparaît que les catégories qu'on lui attribue classiquement : *espace de configuration*, *espace de phases*, *formalisme lagrangien*, *formalisme hamiltonien*, le sont ; ceci simplement parce qu'elles ne possèdent pas la covariance requise; en d'autres termes, parce qu'elles sont en contradiction avec la *relativité galiléenne*. . . ".[5]

With Lagrange's method, in comparison with the usual Hamiltonian formalism, the purely kinetic term of the Hamiltonian $p^2/2m$ is absorbed in the manifold of inertial motion and its symplectic structure.

[5]Free translation: "Analytical mechanics is not an outdated theory; but it appears that the categories classically attributed to it: configuration space, phase space, Lagrangian formalism, Hamiltonian formalism, are (*outdated theories*); this is simply because they do not possess the required covariance; in other words, because they are in contradiction with Galilean relativity. . . "

That is why the Newton's equations, in the above form (L), involve only the potential of the external force.

Note 4. Now it is easy to imagine more sophisticated situations, for example considering geodesics of the sphere as inertial motions, or the initial Lagrange construction with Keplerian motions as inertial motions etc. This Lagrange approach solves the question of equivariance of the dynamics by avoiding the reference to configuration or phase spaces, except maybe in the preliminary construction of the inertial motions which are at the foundation of the method.

77. Deployment of the perturbation

One can deploy the perturbation along the time, that is, consider a new dynamical system defined by a pre-symplectic structure ω on $\mathcal{Y} = \mathcal{M}_0 \times \mathbf{R}$, with

$$\omega = \omega_0 \ominus d[\Omega dt] = \omega_0 \ominus d\Omega \wedge dt,$$

where ω_0 denotes now the symplectic form on the space $\mathcal{M}_0$ of inertial motions. Let $y = (\mu, t)$ a current point on $\mathcal{Y}$ with δy and $\delta' y$ in $\mathrm{T}_y\mathcal{Y} = \mathrm{T}_\mu\mathcal{M}_0 \times \mathrm{T}_t\mathbf{R}$. The evaluation of ω writes:

$$\begin{aligned}
\omega(\delta y, \delta' y) &= \omega_0(\delta\mu, \delta'\mu) - d\Omega(\delta\mu)\delta' t + d\Omega(\delta'\mu)\delta t \\
&= \omega_0(\delta\mu, \delta'\mu) + \omega_0(\mathrm{grad}(\Omega)\delta' t, \delta\mu) \\
&- \omega_0(\mathrm{grad}(\Omega)\delta t, \delta'\mu) \\
&= \omega_0(\delta\mu - \mathrm{grad}(\Omega)\delta t, \delta'\mu - \mathrm{grad}(\Omega)\delta' t)
\end{aligned}$$

The 2-form ω on $\mathcal{Y}$ is clearly presymplectic, its kernel is 1-dimensional and given by:

$$\ker(\omega_y) = \{\delta y \in \mathrm{T}_y\mathcal{Y} \mid \delta\mu = \mathrm{grad}(\Omega)\delta t\}.$$

The point to clear here is the meaning of $\mathrm{grad}(\Omega)$. First of all, the potential Ω is function of $y = (\mu, t)$, as we have seen. Then at the point $y = (y, t)$, $\mathrm{grad}(\Omega)$ denotes the gradient, with respect to ω_0, of the function $\mu \mapsto \Omega(\mu, t)$ defined on $\mathcal{M}_0$. We find here a version of Souriau's construction of the space of motions of a dynamical system

[**Sou70**], but built over the inertial motions instead of built on a phase space.

Now, the quotient of $\mathcal{Y}$ by the characteristic foliation of ω is a smooth manifold $\mathcal{M}$. Indeed, the distribution $y \mapsto \ker(\omega_y)$ is transverse to the slices $\mathcal{M}_0 \times \{t\}$ and the restrictions π_t of the projection $\pi : \mathcal{Y} \to \mathcal{M}$ to $\mathcal{M}_0 \times \{t\}$ form an atlas of $\mathcal{M}$. Moreover, since ω is closed, denoting by ξ the vector field $\xi(y) = (\mathrm{grad}(\Omega), 1)$ generating $\ker(\omega_y)$, we get $\pounds_\xi(\omega) = 0$, which implies that there exists a closed 2-form, denoted by the same letter ω, on $\mathcal{M}$ such that its pullback on $\mathcal{Y}$ by π is ω. Of course, if the characteristic flow ξ is complete, that is, if the second projection $\mathrm{pr}_2 : \mathcal{Y} \to \mathbf{R}$, restricted to every integral curve, is surjective, then $(\mathcal{M}, \omega)$ is symplectomorph to $(\mathcal{M}_0, \omega_0)$.

78. Example of the oscillator

We consider a couple of points $x = (x_1, x_2) \in \mathbf{R}^3 \times \mathbf{R}^3$ and the Newton's equations

$$\frac{d^2x(t)}{dt^2} = \begin{pmatrix} x_2(t) - x_1(t) \\ x_1(t) - x_2(t) \end{pmatrix}$$

(A) **The symplectic form on inertial motions** — A variation of an inertial motion $\mu \in \mathcal{M}_0$ writes necessarilly

$$\delta\mu = [t \mapsto \delta X + t\delta v] \in T_\mu\mathcal{M}_0.$$

Denoting by $\dot\mu$ the speed of the motion we can check that, for two such variations $\delta\mu$ and $\delta'\mu$, the value

$$\omega(\delta\mu, \delta'\mu) = m\left[\langle\delta\dot\mu(t), \delta'\mu(t)\rangle - \langle\delta'\dot\mu(t), \delta\mu(t)\rangle\right] \qquad (\spadesuit)$$

does not depend on the instant t where it is computed, and it is equal to the expression given in ($\clubsuit$) above.

(B) **The action of Galilean group** — The action of Galilean group assumes a space-time $E = \mathbf{R}^3 \times \mathbf{R}$, the points of E are denoted by (x, t). The Galilean group is the following group of matrices with

$A \in SO(3)$, $b, c \in \mathbf{R}^3$ and $e \in \mathbf{R}$:

$$g = \begin{pmatrix} A & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + bt + c \\ t + e \\ 1 \end{pmatrix},$$

where E is imbedded in this affine picture as the level 1 of $E \times \mathbf{R}$. Now, an inertial motion is an affine line in E, that is :

$$\mu = \left\{ \begin{pmatrix} x + tv \\ t \end{pmatrix} \in E \mid t \in \mathbf{R} \right\},$$

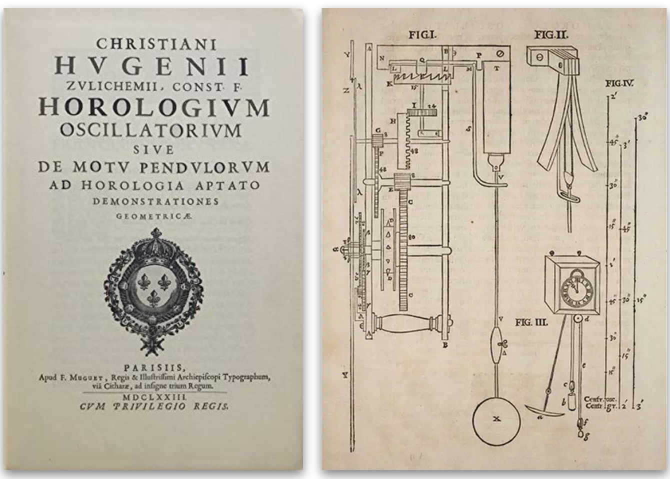
where, as usual $x, v \in \mathbf{R}^3$. Then, Galilean group acting on E maps the affine line μ into another affine line $g(\mu)$ and immediately :

$$g(\mu) = \left\{ \begin{pmatrix} Ax + c + t(Av + b) \\ t + e \end{pmatrix} \in E \mid t \in \mathbf{R} \right\},$$

or again:

$$g(\mu) = [t \mapsto Ax + c + (t - e)(Av + b)].$$

The symplectic form ω on $\mathcal{M}_0$ is obviously invariant by g, indeed, let $\mu' = g(\mu)$, then $\delta\mu' = [t \mapsto A\delta x + (t - e)A\delta v]$. Computed for $t = e$, the expression (♠) above gives $g^*(\omega) = \omega$, since $A \in SO(3)$.



Horologium Oscillatorium by Christiaan Huygens

Poisson Bracket in Diffeology

In this note we show how to understand the Poisson bracket in diffeology, working directly on the group of Hamiltonian diffeomorphisms without involving tangent spaces and Hamiltonian gradients. Indeed, as it is usually defined, Poisson bracket seems to be a contravariant object and therefore not really adapted to diffeology, but it can be defined in a covariant way, which is more adequate with the diffeology framework.

The question of Poisson bracket in diffeology has been raised many times, and recently again by Jim Stasheff in a private discussion. This time I want to give an answer that I find satisfactory.

79. Classical Poisson bracket

The Poisson bracket is generally introduced in symplectic geometry as a binary operation on the space of functions. Let (M, ω) be a symplectic manifold, that is,

$$\omega \in \Omega^2(\mathrm{M}), \quad d\omega = 0 \quad \text{and} \quad \ker(\omega) = 0.$$

Let $x \mapsto u$ be a smooth real function on M, we denote by

$$\mathrm{grad}_\omega(u) = \omega^{-1}(du)$$

is *symplectic gradient*. Here ω is regarded as a linear isomorphism from TM to $\mathrm{T}^*\mathrm{M}$, so $\omega^{-1}(du)$ is a tangent vector field. The Poisson bracket is usually defined in the following way [**Sou70**]:

303. The classical definition Let $x \mapsto u$ and $x \mapsto v$ be two smooth real functions on M,[1] The *Poisson bracket* of u and v is denoted and defined by:[2]

$$\{u, v\} = \omega(\mathrm{grad}_\omega(u), \mathrm{grad}_\omega(v)).$$

The Poisson bracket as it is so defined is a bilinear map

$$\{\cdot,\cdot\} : \mathcal{C}^\infty(\mathrm{M}, \mathbf{R})^2 \to \mathcal{C}^\infty(\mathrm{M}, \mathbf{R})$$

That satisfies some classical relations called Jacobi identity we do not discuss here.

Next, by applying the definition above of the symplectic gradient, we have also the equivalent definition:

$$\{u, v\} = du(\mathrm{grad}_\omega(v)) \quad \text{also denoted by} \quad \frac{\partial u}{\partial x}(\mathrm{grad}_\omega(v)).$$

80. Poisson bracket in diffeology

The problem with all these concept involving tangent vector fields and Lie algebras is that they have not only one interpretation in diffeology. That is why we have to bypass as much as possible the introduction of tangent spaces in their generalizations in diffeology. That works for the moment map, relatively well, as the many examples in [**PIZ10**] have shown. This is what we propose in the case of the Poisson bracket, to get a definition that covers the classical case and satisfies the constraints of a good diffeological equivalent.

Let us now come back to the classical picture for a while. Assume that the vector field

$$x \mapsto \mathrm{grad}_\omega(u)$$

is integrable. That is, it defines a 1-parameter group of diffeomorphisms

$$t \mapsto e^{\mathrm{grad}_\omega(u)}.$$

---

[1] They are called *dynamic variables* by Souriau.

[2] It is denoted by $[u, v]_\mathrm{P}$ in [**Sou70**].

Let us change our notation to:

$$\mathrm{Z_M} : x \mapsto \mathrm{Z_M}(x) = \mathrm{grad}_\omega(u) \quad \text{and} \quad \mathrm{Z'_M} : x \mapsto \mathrm{Z'_M}(x) = \mathrm{grad}_\omega(u').$$

We have

$$\{u, u'\} = \omega_x(\mathrm{Z_M}(x), \mathrm{Z'_M}(x))$$

Assume now that Z, Z' belong to the Lie algebra $\mathcal{G}$ of a Lie group G

$$\mathrm{Z}, \mathrm{Z}' \in \mathcal{G}.$$

The 1-parameter groups $\{e^{t\,\mathrm{grad}_\omega(u)}\}_{t\in\mathbf{R}}$ and $\{e^{t\,\mathrm{grad}_\omega(u')}\}_{t\in\mathbf{R}}$ are just the action of two 1-parameter group in G

$$\{e^{t\mathrm{Z}}\}_{t\in\mathbf{R}} \quad \text{and} \quad \{e^{t\mathrm{Z}'}\}_{t\in\mathbf{R}} \quad \text{belong to} \quad \mathrm{Hom}^\infty(\mathbf{R}, \mathrm{G}),$$

and $\mathrm{Z_M}(x)$ and $\mathrm{Z'_M}(x)$ are the fundamental vector fields associated with Z and Z′ in $\mathcal{G}$. Thus,

$$\omega_x(\mathrm{Z_M}(x), \mathrm{Z'_M}(x)) = \hat{x}^*(\omega)(\mathrm{Z}, \mathrm{Z}'),$$

where

$$\hat{x} : g \mapsto g_\mathrm{M}(x)$$

is the *orbit map* at the point x and $g_\mathrm{M} \in \mathrm{Diff}(\mathrm{M})$ denotes the action of G on M. Indeed

$$\begin{aligned} \hat{x}^*(\omega)(\mathrm{Z}, \mathrm{Z}') &= \omega_x(\mathrm{D}(\hat{x})_{1_\mathrm{G}}(\mathrm{Z}), \mathrm{D}(\hat{x})_{1_\mathrm{G}}(\mathrm{Z}')) \\ &= \omega_x(\mathrm{Z_M}(x), \mathrm{Z'_M}(x)), \end{aligned}$$

because

$$\mathrm{Z_M}(x) := \left.\frac{\partial e^{t\mathrm{Z}}_\mathrm{M}(x)}{\partial t}\right|_{t=0} = \mathrm{D}(\hat{x})_{1_\mathrm{G}}(\mathrm{Z}),$$

where 1_G denotes the identity in G. Now,

304. Proposition. If the group G preserves ω, that is, if

$$\forall g \in \mathrm{G}, \quad g^*_\mathrm{M}(\omega) = \omega,$$

then

$$\forall g \in \mathrm{G}, \quad \mathrm{L}(g)^*(\hat{x}^*(\omega)) = x^*(\omega).$$

where $\mathrm{L}(g)$ is the left multiplication in G. Thus the map $x \mapsto \hat{x}^*(\omega)$ defined on M takes its values in the vector space of left invariant 2-forms on G.

Let us denote for now

$$\mathcal{G}_k^* = \{\varepsilon \in \Omega^k(G) \mid \forall g \in G, \quad L(g)^*(\varepsilon) = \varepsilon\}.$$

Thus,

$$\{\cdot,\cdot\} = [x \mapsto \hat{x}^*(\omega)] \in \mathcal{C}^\infty(X, \mathcal{G}_2^*).$$

Such that

$$\{\cdot,\cdot\}_x(Z, Z') = \omega_x(Z_M(x), Z'_M(x)).$$

Come back to our symplectic gradients: the flow $\{e^{t\,\mathrm{grad}_\omega(u)}\}_{t\in\mathbf{R}}$ they define is called *Hamiltonian* because its Hamiltonian [**TB**, § 9.15] is $[x \mapsto u]$. Of course the group

$$H_\omega = \mathrm{Ham}(M, \omega)$$

is not a Lie group, but it is a diffeological group and as such obeys to all diffeological constructions, in particular the diffeological vector space of left-invariant k-forms on H_ω is well defined

$$\mathcal{H}_{\omega,k}^* = \{\varepsilon \in \Omega^k(H_\omega) \mid L(g)^*(\varepsilon) = \varepsilon\},$$

for any diffeological space X equipped with a closed 2-form ω. The space $\mathcal{H}_\omega^* = \mathcal{H}_{\omega,1}^*$ has been already defined as the space of *momenta* of the group H_ω [**TB**, § 7.12].

Therefore, we can define now the Poisson bracket in full generality:

305. The Poisson bracket in diffeology. Let (X, ω) be a parasymplectic space. Let $\mathrm{Ham}(X, \omega)$ be the group of Hamiltonian diffeomorphisms [**TB**, § 9.15], and $\mathcal{H}_{\omega,2}^*$ be the diffeogical vector space of left-invariant 2-form on X. We call *Poisson bracket* the map

$$\{\cdot,\cdot\} : x \mapsto \hat{x}^*\omega, \quad \{\cdot,\cdot\} \in \mathcal{C}^\infty(X, \mathcal{H}_{\omega,2}^*).$$

But this definition coincides with the classical version when (X, ω) is a symplectic manifold.

☞ Proof. Consider a n-plot $P : U \to H_\omega$ with $P : r \mapsto g_r$, centered at $1_G = P(0)$. Then, for all $x \in X$, for all $r \in U$ and $\delta r, \delta' r \in \mathbf{R}^n$:

$$\begin{aligned}\{\cdot,\cdot\}_x(P)_r(\delta r, \delta' r) = \hat{x}^*\omega(P)_r(\delta r, \delta' r) &= \omega(\hat{x} \circ P)_r(\delta r, \delta' r) \\ &= \omega(r \mapsto g_r(x))_r(\delta r, \delta' r).\end{aligned}$$

Since $\{\cdot,\cdot\}_X$ is left-invariant it is defined by its value at the identity [**TB**, § 6.40, 7.18]. Thus, let us compute $\{\cdot,\cdot\}_X(\mathrm{P})$ at $r = 0$ for two vectors $v, v' \in \mathbf{R}^n$.

$$\begin{aligned}
\omega(r \mapsto g_r(x))_0(v, v') &= \omega(r \mapsto g_r(x))_0(v, v') \\
&= \omega_{g_0(x)}\left(\left.\frac{\partial g_r(x)}{\partial r}\right|_{r=0}(v), \left.\frac{\partial g_r(x)}{\partial r}\right|_{r=0}(v') \right) \\
&= \omega_{\mathrm{X}}(\mathrm{Z}_{\mathrm{M}}(\mathrm{x}), \mathrm{Z}'_{\mathrm{M}}(\mathrm{x})),
\end{aligned}$$

where

$$\begin{aligned}
\left.\frac{\partial g_r(x)}{\partial r}\right|_{r=0}(v) &= \mathrm{D}(s \mapsto g_{sv}(x))_{s=0}(1), \quad (s \in \mathbf{R}) \\
&= \mathrm{D}(\hat{\mathrm{x}})_{1_{\mathrm{G}}}\left(\left.\frac{\partial g_{sv}}{\partial s}\right|_{s=0} \right) \\
&= \mathrm{D}(\hat{\mathrm{x}})_{1_{\mathrm{G}}}(\mathrm{Z}),
\end{aligned}$$

with

$$\mathrm{Z} = \left.\frac{\partial g_{sv}}{\partial s}\right|_{s=0}.$$

Indeed $s \mapsto g_{sv}$ is a centered path at the identity 1_{G}, then its derivative defines a tangent vector $\mathrm{Z} \in \mathrm{T}_{1_{\mathrm{G}}}(\mathrm{G})$, that is an element of the Lie algebra $\mathcal{G}$. If the action of G on M is Hamiltonian then $\mathrm{Z} = \mathrm{grad}_\omega(u)$ and $\mathrm{Z}' = \mathrm{grad}_\omega(u')$ Therefore:

$$\omega(r \mapsto g_r(x))_0(v, v') = \omega_{\mathrm{X}}(\mathrm{Z}_{\mathrm{M}}(x), \mathrm{Z}'_{\mathrm{M}}(x)) = \{u, u'\},$$

which is the ordinary definition of the Poisson bracket in symplectic geometry. ▶

<u>**306. Remark**</u> The Poisson bracket is not only a map from X to left-invariant 2-forms on $\mathrm{Ham}(\mathrm{X}, \omega)$:

(A) The Poisson bracket takes its values in the space of invariant closed 2-forms:

$$d[\{\cdot,\cdot\}(x)] = d[\hat{x}^*\omega] = \hat{x}^*[d\omega] = 0 \quad \text{for all } x \in \mathrm{X}.$$

Developed, this gives the <u>Jacobi identity</u>.

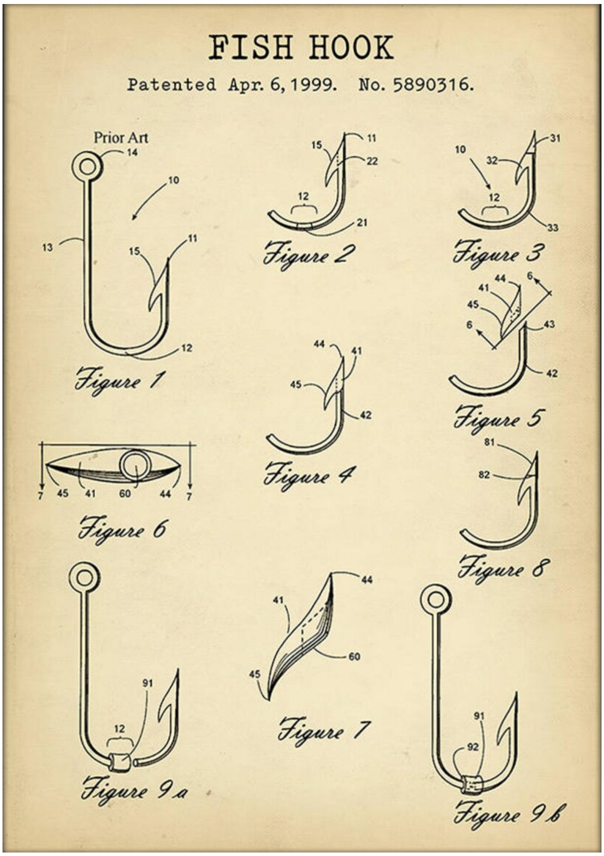
(B) Consider the special case when ω is exact and its primitive is also exact:

$$\omega = d\alpha \quad \text{and} \quad h^*(\alpha) = \alpha \quad \text{for all } h \in \mathrm{Ham}(X, \omega).$$

Then, the Poisson bracket is related to the universal moment map μ_ω [TB, §9.14] :

$$\{\cdot, \cdot\}(x) = d[\mu_\omega(x)] \quad \text{for all } x \in X \quad ,$$

since $\mu_\omega(x)$ is in this case simply $\hat{x}^*(\alpha)$.



Fish Hook : Crochets de Poisson ?

Smooth Embeddings and Smoothly Embedded Subsets

Since Henri Joris paper we potentially know that, paradoxically, the semi-cubic $y^2 - x^3 = 0$ is an embedded submanifold of $\mathbf{R}^2$. So, how to understand the singularity of the cusp? This is because, if the cusp is embedded, it is not smoothly embedded. And this note will detail this aspect.

81. Smooth embeddings

The semi-cubic is the subset $\mathfrak{S} \subset \mathbf{R}^2$ of equation $y^2 - x^3 = 0$. It is represented in Figure 43. It is the image of the map

$$j : \mathbf{R} \to \mathbf{R}^2, \quad j : t \mapsto \begin{pmatrix} t^2 \\ t^3 \end{pmatrix}.$$

The map j is not an immersion since $j'(0) = 0$, where j' is the derivative of j. But a longtime question was : is j an induction?

I recall that the map j is an induction [**TB**, § 1.29] if and only if: j is smooth, injective and the inverse $j^{-1} : \mathfrak{S} \to \mathbf{R}$ is smooth, where $\mathfrak{S} = \mathrm{val}(j)$ is equipped with the subset diffeology.

In that case, that is equivalent to the following:

Question. For any smooth parametrization $\mathrm{P} : r \mapsto (x(r), y(r))$ with values in $\mathfrak{S}$, that is, $y(r)^2 - x(r)^3 = 0$ for all r, is the following map smooth?

$$t(r) = \frac{y(r)}{x(r)}$$

The solution to this question comes from a theorem due to Henri Joris [Jor82], which was pointed out to me by Yael Karshon.[1] In a simple version:

307. Theorem (H. Joris). *If a real map f is such that $t \mapsto f(t)^2$ and $t \mapsto f(t)^3$ are smooth, then f is smooth.*

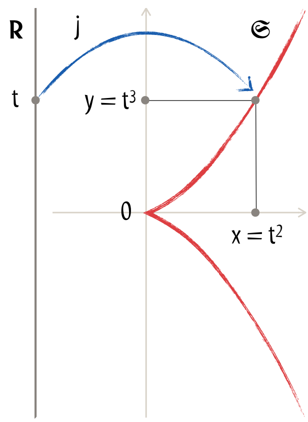


Figure 43. The semi-cubic $y^2 = x^3$.

308. Corollary. *The map $j : t \mapsto (t^2, t^3)$ is an induction from $\mathbf{R}$ into $\mathbf{R}^2$. The semi-cubic $\mathfrak{S}$ image of j is an embedded submanifold of $\mathbf{R}^2$.*

Proof. Clearly $j : t \mapsto (x = t^2, y = t^3)$ is smooth, and injective: $t = \sqrt[3]{y}$. Let $r \mapsto \mathrm{P}(r) = (x(r), y(r))$ be a plot in $\mathbf{R}^2$ with value in $\mathfrak{S}$. Then, $j^{-1} \circ \mathrm{P}(r) = t(r)$ such that $r \mapsto t(r)^2$ and $r \mapsto t(r)^3$ are smooth. To apply Joris theorem we need to come back to a map from $\mathbf{R}$ to $\mathbf{R}$. We can use Boman's theorem][Bom67]

309. Theorem (J. Boman). *A continuous parametrization $\mathrm{P} : \mathrm{U} \to \mathbf{R}^m$, is smooth if and only if, for any smooth path γ in U, the composite $\mathrm{P} \circ \gamma$ is smooth.*

[1] In a private discussion, but since commented in [KMW22].

But, for each such smooth path, the composite $s \mapsto t(r(s))$ is smooth, thanks indeed to Joris theorem. Thus, $r \mapsto \mathrm{P}(r)$ is smooth. To finish j is an embedding because $y \mapsto \sqrt[3]{y}$ is an homeomorphism of $\mathbf{R}$. ▶

So, we have a figure, the semi-cubic, image of $\mathbf{R}$ by an induction, which is by construction a submanifold of $\mathbf{R}^2$ because its subset diffeology is equivalent to $\mathbf{R}$.

There is clearly something weird in that situation, the cusp at $(0,0)$ is obviously a singularity. But since it is transparent to the subset diffeology, how to capture it? That is the question.

82. Smoothly embedded subsets

The answer lies in the relationship between the ambient space $\mathbf{R}^2$ and the subspace $\mathfrak{S}$. Consider the pseudo-group of diffeomorphisms of $\mathbf{R}^2$ that preserve $\mathfrak{S}$, that is, that fix globally $\mathfrak{S}$,

$$\mathrm{Diff}_{\mathrm{loc}}(\mathbf{R}^2, \mathfrak{S}) = \{\phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbf{R}^2) \mid \phi(\mathfrak{S}) \subset \mathfrak{S}\}$$

Since ϕ is a local diffeomorphism $\phi(\mathfrak{S}) = \phi(\mathfrak{S} \cap \mathrm{dom}(\phi))$. If $\mathfrak{S} \cap \mathrm{dom}(\phi)$ is empty, there is nothing to check. Then,

310. Proposition. Every local diffeomorphism $\phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbf{R}^2, \mathfrak{S})$ fixes the point $(0,0)$,

$$\forall \phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbf{R}^2, \mathfrak{S}),\ \phi(0,0) = (0,0).$$

☞ Proof. Let $\phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbf{R}^2, \mathfrak{S})$ such that $\phi(0,0) = (x,y)$ and $(x,y) \neq (0,0)$. The restriction $\phi \restriction \mathfrak{S}$ belongs to $\mathrm{Diff}_{\mathrm{loc}}(\mathfrak{S})$. Thus, $\varphi = j^{-1} \circ (\phi \restriction \mathfrak{S}) \circ j$ belongs to $\mathrm{Diff}_{\mathrm{loc}}(\mathbf{R})$.

$$\begin{array}{ccc} \mathbf{R} & \xrightarrow{\ j\ } & \mathfrak{S} \subset \mathbf{R}^2 \\ \varphi \uparrow & & \uparrow \phi \restriction \mathfrak{S} \\ \mathbf{R} & \xrightarrow[\ j\ ]{} & \mathfrak{S} \subset \mathbf{R}^2 \end{array}$$

Let us derive $j \circ \varphi = (\phi \upharpoonright \mathfrak{S}) \circ j$ at the point 0, and let $j(t) = (x, y) = \phi(0, 0)$:

$$\begin{aligned} \mathrm{D}(j \circ \varphi)(0) &= \mathrm{D}((\phi \upharpoonright \mathfrak{S}) \circ j)(0) \\ \mathrm{D}(j)(t) \circ \mathrm{D}(\varphi)(0) &= \mathrm{D}(\phi \upharpoonright \mathfrak{S})(0,0) \circ \mathrm{D}(j)(0) \\ \mathrm{D}(j)(t) \circ \mathrm{D}(\varphi)(0) &= 0. \end{aligned}$$

But $t \neq 0$ implies $\mathrm{rank}(\mathrm{D}(j)(t)) = 1$, and $\mathrm{D}(\varphi)(0)$ is a non-zero number, thus the composite $\mathrm{D}(j)(t) \circ \mathrm{D}(\varphi)(0)$ cannot be zero. ▶

So, although the local diffeomorphisms are transitive on $\mathfrak{S}$, the local diffeomorphisms of $\mathbf{R}^2$ preserving $\mathfrak{S}$ are not. This is this iatus which capture the singularity at 0. This remark leads then to a refinement of the concept of embedding. Let X be a diffeological space and $\mathfrak{S} \subset \mathrm{X}$ an embedded subset [TB, § 2.13].

Let us begin by defining two groupoids:

(A) Let $\mathrm{K}_{\mathfrak{S}}$ be the groupoid of germs of local diffeomorphisms of $\mathfrak{S}$.

$$\begin{cases} \mathrm{Obj}(\mathrm{K}_{\mathfrak{S}}) = \mathfrak{S} \\ \mathrm{Mor}_{\mathfrak{S}}(x, x') = \{\mathrm{germ}(\phi)_x \mid \phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathfrak{S}),\ \phi(x) = x'\} \end{cases}$$

(B) Let $\mathrm{K}_{\mathrm{X},\mathfrak{S}}$ be the groupoid of germs of local diffeomorphisms of X preserving $\mathfrak{S}$.

$$\begin{cases} \mathrm{Obj}(\mathrm{K}_{\mathrm{X},\mathfrak{S}}) = \mathfrak{S} \\ \mathrm{Mor}_{\mathrm{X},\mathfrak{S}}(x, x') = \{\mathrm{germ}(\phi)_x \mid \phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathrm{X}, \mathfrak{S}),\ \phi(x) = x'\} \end{cases}$$

There is a natural morphism restriction Φ from $\mathrm{K}_{\mathrm{X},\mathfrak{S}}$ to $\mathrm{K}_{\mathfrak{S}}$:

$$\Phi_{\mathrm{Obj}} = 1_{\mathfrak{S}} \quad \text{and} \quad \Phi_{\mathrm{Mor}} : \mathrm{germ}(\phi)_x \mapsto \mathrm{germ}(\phi \upharpoonright \mathfrak{S})_x.$$

That leads to the following definition:

<u>311. Definition 1.</u> We shall say that an embedded subset $\mathfrak{S} \subset \mathrm{X}$ is *smoothly embedded* if the morphism Φ, from $\mathrm{K}_{\mathrm{X},\mathfrak{S}}$ to $\mathrm{K}_{\mathfrak{S}}$ is surjection on the arrows. That is, if Φ is a full functor.

In other words, the embedded subset $\mathfrak{S}$ is smoothly embedded if it is embedded and the germ of any local diffeomorphism of $\mathfrak{S}$ can be extended to a local diffeomorphism of X; or again if the germ of any local diffeomorphism of $\mathfrak{S}$ is the imprint, or the trace, of a local diffeomorphism of X.

Thus, the semi-cubic is indeed and embedded submanifold, but not smoothly embedded.

This notion of embedded subset goes back to the embeddings. We get the definition of smooth embeddings.

312. Definition 2. Let $j : \Sigma \to X$ be an embedding. we shall say that j is a smooth embedding if $\mathfrak{S} = j(\Sigma)$ is smoothly embedded in X.

Conclusion: The injection $j : t \mapsto (t^2, t^3)$ is an embedding, but not a smooth embedding.

Notes

Seifert Orbifolds

The quotient of a 3-manifold by an effective action of S^1, without fixed points, is a classical example of orbifold. But how does that fit within the diffeological framework?

We'll talk about classical things and construction in differential geometry, but from the point of view of diffeology. We'll see how, in this example, the vocabulary and constructions of diffeology fit the needs of the problem. We shall consider in particular the diffeological definition of an orbifold, that is, a diffeological space locally diffeomorphic to a quotient $\mathbf{R}^n/\Gamma$, where Γ is a finite subgroup of the linear group $\mathrm{GL}(n, \mathbf{R})$, see [**IKZ10**] for the details.

Warning. One should not confuse what we call in this note a "Seifert orbifold" with what topologists call "Seifert fibered orbifold", as it appears in [**BS85**] for example. For us the orbifold is the space of Seifert fibers of a Seifert fibered manifold, and not an orbifold that would be the total space of a Seifert fibration.

83. A little bit of smooth Lie group actions

Consider a smooth action of a Lie group G on a manifold M.

(A) Diffeologically speaking, such an action is a smooth homomorphism $g \mapsto g_{\mathrm{M}}$, from G to Diff(M), where Diff(M) is equipped with the functional diffeology [**TB**, § 7.4].

(B) Let $x \in M$ and $H = St(x)$, the stabilizer of x. The orbit map $g \mapsto g_M(x)$ from G to M is strict, that means that the projection $\mathrm{class}(g) \mapsto g_M(x)$, defined on the coset G/H into M is an induction. In other words, the map class : $G/H \to \mathcal{O}_x$ is a diffeomorphism, where G/H is equipped with the quotient diffeology and $\mathcal{O}_x \subset M$ is equipped with the subset diffeology [**IZK12**, § 1]: equipped with the subset diffeology, the orbit $\mathcal{O}_x$ is a manifold diffeomorphic to the coset G/H. Now, if G is compact, then this induction is an embedding and the orbit is an embedded submanifold.

(C) If G is compact, the type of stabilizers (or orbits) of a smooth action of G on a manifold M, is given by the *Theorem of Principal Orbits* [**Bre72**, Theorem 3.1 of Part IV].

313. Principal orbits. There exists a maximum orbit type G/H for G on M (i.e., H is conjugate to a subgroup of each isotropy group). The union $M_{(H)}$ of the orbits of type G/H is open and dense in M.

In particular, the stabilizers of any two points in $M_{(H)}$ are conjugate. The orbits of points in $M_{(H)}$ are called *principal orbits*.

Non-principal orbits are called *singular orbits*. If a singular orbit has the same dimension than the principal orbits, the orbit is said to be *exceptional*.

(D) If G is compact, the Theorem 2.4 of part VI of *Compact Group Action on Manifolds* [**Bre72**] states that:

314. Smooth linear tube. Let x be any point in M, and let H be the stabilizer of x. There exists a vector space V, an orthogonal action of H on V, and a local G invariant diffeomorphism $\varphi : G \times_H V \to M$.

The space $G \times_H V$ is the quotient of $G \times V$ by the diagonal action of H, $h_{G\times V} : (g, v) \mapsto (gh^{-1}, h_V(v))$, where $h \mapsto h_V$ denotes the action of H on V.

The image of the local diffeomorphism φ is an open invariant tube about the orbit $\mathcal{O}_x$, image by φ of the zero-section in $G \times_H V$.

84. Circle actions on manifolds

Consider a smooth action of the circle S^1 on a manifold M. According to the Theorem of Principal Orbits of compact groups actions on manifolds, there exists a S^1-invariant open dense subset of M such that all of its points have the same principal stabilizer H, a closed subgroup of S^1. And this principal stabilizer is contained in every stabilizer.

If the principal stabilizer is S^1 then all stabilizers are S^1 and the action is trivial.

If the action is not trivial then H is a cyclic group $\mathbf{Z}_m = \{\varepsilon \mid \varepsilon^m = 1\}$.

In the case of a non trivial action of S^1, it is always possible to consider the quotient group S^1/H, which is isomorphic to S^1, and the situation is reduced to an effective action of S^1, that is, an action with principal stabilizer $\{1\}$. This will be what we assume now.

If the action has no fixed points then the singular orbits are exceptional, and conversely.

85. Seifert orbifolds

Now, we can present the object of our note. We refer to [**Sco83**] for the vocabulary and general context.

315. Seifert fibration. *A* Seifert Fibered Space *(or Seifert fibration) is a* 3*-manifold* M *with an effective action of* S^1 *without fixed points.*

Because all the stabilizers are cyclic, the *fibers* of the Seifert Fibered Space, that is, the orbits of the action of the circle, are diffeomorphic to the circle, when equipped with their subset diffeology [**IZK12**, § 1]. What about the quotient space?

316. Seifert orbifolds. Let M be a 3-manifold with an effective action of S^1, without fixed points. Then, the quotient space $Q = M/S^1$ is a 2-manifold if the action is principal, or a 2-orbifold with isolated conic singularities otherwise.

☞ Proof. Thanks to the Linear Tube Theorem, every orbit $\mathcal{O}_x$ has an equivariant open neighborhood diffeomorphic to a linear tube of type $S^1 \times_{Z_m} \mathbf{C}$, where $\mathbf{Z}_m$ is the stabilizer of x. Then, about the orbit $\mathcal{O}_x \in Q$, the quotient space Q is locally diffeomorphic to $[S^1 \times_{Z_m} \mathbf{C}]/S^1$, where the action of S^1 is given by $\tau\, \mathrm{class}(z, Z) = \mathrm{class}(\tau z, Z)$. Note in particular that $\mathrm{class}(z, Z) = z\, \mathrm{class}(1, Z)$.

Consider the map $J : \mathbf{C} \to S^1 \times_{Z_m} \mathbf{C}$, defined by $J(Z) = \mathrm{class}(1, Z)$. The map J is an induction. Indeed, it is clearly injective. Moreover, let $r \mapsto \zeta(r)$ be a plot in $J(\mathbf{C}) \subset S^1 \times_{Z_m} \mathbf{C}$. Since $\mathrm{class} : S^1 \times \mathbf{C} \to S^1 \times_{Z_m} \mathbf{C}$ is a subduction, by construction, there exists always plots $r \mapsto (z(r), Z(r))$ in $S^1 \times \mathbf{C}$ such that, locally, $\zeta(r) = \mathrm{class}(z(r), Z(r))$. But since $\zeta(r) \in J(\mathbf{C})$, for all r in the domain of the plot there is $Z' \in \mathbf{C}$ such that $\mathrm{class}(z(r), Z(r)) = \mathrm{class}(1, Z')$. Thus $z(r) \in \mathbf{Z}_m \subset \mathbf{C}$, and since $\mathbf{Z}_m \subset \mathbf{C}$ is diffeologically discrete, $r \mapsto z(r)$ is locally constant, $z(r) = \varepsilon \in \mathbf{Z}_m$. Hence, $\zeta(r) =_{\mathrm{loc}} \mathrm{class}(1, \varepsilon Z(r))$, with $r \mapsto \varepsilon Z(r)$ smooth. Therefore, J is an induction.

Next, every S^1-orbit in $S^1 \times_{Z_m} \mathbf{C}$ writes $S^1 \cdot \mathrm{class}(1, Z) = \{\mathrm{class}(z, Z) \mid z \in S^1\}$, for some $Z \in \mathbf{C}$. Its intersection with $J(\mathbf{C})$ is the set

$$\left(S^1 \cdot \mathrm{class}(1, Z)\right) \cap J(\mathbf{C}) = \{\mathrm{class}(1, \varepsilon Z) = J(\varepsilon Z) \mid \varepsilon \in \mathbf{Z}_m\}$$

Thus, there exists a natural bijection $j : \mathbf{C}/\mathbf{Z}_m \to \left(S^1 \times_{Z_m} \mathbf{C}\right)/S^1$ mapping every $\mathbf{Z}_m$-orbit in $\mathbf{C}$ to the corresponding S^1-orbit in $S^1 \times_{Z_m} \mathbf{C}$.

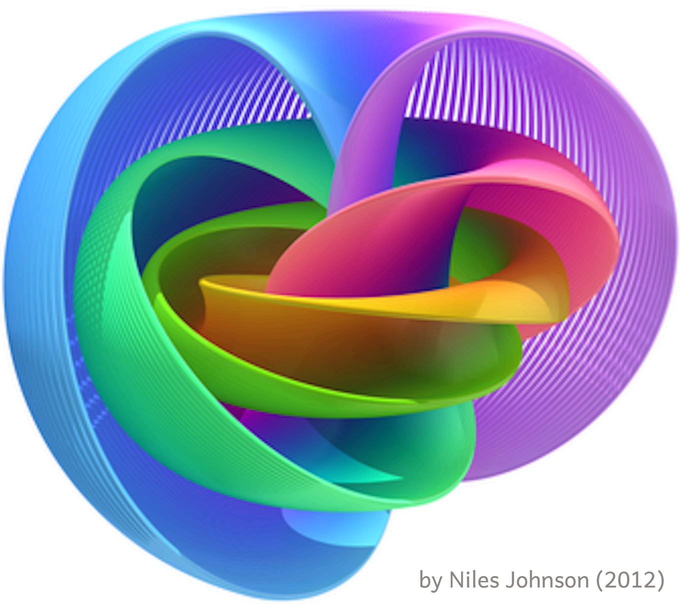
$$j(\mathrm{class}(Z)) = S^1 \cdot \mathrm{class}(1, Z).$$

Now, let $\mathrm{pr} : S^1 \times_{Z_m} \mathbf{C} \to \left(S^1 \times_{Z_m} \mathbf{C}\right)/S^1$ be the natural projection. Let us prove that the restriction $\mathrm{pr} \restriction J(\mathbf{C})$ is a still a subduction. Let $r \mapsto \zeta(r)$ be a plot in $\left(S^1 \times_{Z_m} \mathbf{C}\right)/S^1$, locally $\zeta(r) = \mathrm{class}(z(r), Z(r))$, where $r \mapsto z(r)$ and $r \mapsto Z(r)$ are smooth. But, $\mathrm{class}(z(r), Z(r)) = \bar{z}(r) \cdot \mathrm{class}(1, z(r)Z(r))$, then $\mathrm{pr}(\mathrm{class}(z(r), Z(r))) = \mathrm{pr}(\mathrm{class}(1, z(r)Z(r)))$, and $\mathrm{class}(1, z(r)Z(r))$ belongs to $J(\mathbf{C})$. Thus, as claimed, $\mathrm{pr} \restriction J(\mathbf{C})$ is still a subduction.

Therefore, since $\mathrm{class} : \mathbf{C} \to \mathbf{C}/\mathbf{Z}_m$ and $\mathrm{pr} \restriction J(\mathbf{C}) : J(\mathbf{C}) \to \left(S^1 \times_{Z_m} \mathbf{C}\right)/S^1$ are two subductions, since J is an induction, and the factorization $j : \mathbf{C}/\mathbf{Z}_m \to \left(S^1 \times_{Z_m} \mathbf{C}\right)/S^1$ is a bijection, j is a diffeomorphism.

$$
\begin{array}{ccccc}
\mathbf{C} & \xrightarrow{\ J\ } & \mathrm{S}^1 \times_{\mathbf{Z}_m} \mathbf{C} & \xrightarrow{\ \phi\ } & \mathrm{M} \\
{\scriptstyle \mathrm{class}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \pi} \\
\mathbf{C}/\mathbf{Z}_m & \xrightarrow[\ j\ ]{} & \left(\mathrm{S}^1 \times_{\mathbf{Z}_m} \mathbf{C}\right)/\mathrm{S}^1 & \xrightarrow[\ f\ ]{} & \mathrm{M}/\mathrm{S}^1
\end{array}
$$

Finally, the equivariant local diffeomorphism $\phi : \mathrm{S}^1 \times_{\mathbf{Z}_m} \mathbf{C} \to \mathrm{M}$, given by the Smooth Linear Tube Theorem above, projects on a local diffeomorphism $f : \left(\mathrm{S}^1 \times_{\mathbf{Z}_m} \mathbf{C}\right)/\mathrm{S}^1 \to \mathrm{M}/\mathrm{S}^1$. The composite $f \circ j$ is then a local diffeomorphism from $\mathbf{C}/\mathbf{Z}_m$ into M/S^1. Therefore, M/S^1 is an orbifold according to [IKZ10, Definition 6]. The singular points are the images by $f \circ j$ of $\mathrm{class}(0) \in \mathbf{C}/\mathbf{Z}_m$. They are clearly isolated and conic. If there is no exceptional orbit, then M/S^1 is obviously just a manifold (which is just a consequence of the Linear Tube Theorem). ▶



A vision of the Hopf fibration

Symplectic Spaces Without Hamiltonian Diffeomorphisms

We consider the 2-dimensional irrational torus $T^2_{\alpha,\beta} = T_\alpha \times T_\beta$, where $T_\alpha = \mathbf{R}/\mathbf{Z} + \alpha\mathbf{Z}$ and $T_\beta = \mathbf{R}/\mathbf{Z} + \beta\mathbf{Z}$, with $\alpha, \beta \in \mathbf{R} - \mathbf{Q}$. We check that 2-form $\omega = \theta_\alpha \wedge \theta_\beta$, where θ_α and θ_β are the projection of dt on T_α and T_β, is symplectic, according to the definition of a symplectic form in diffeology. Then, we show that the group of symplectic transformations $\mathrm{Diff}(T^2_{\alpha,\beta}, \omega)$ has no Hamiltonian transformation.

86. Diffeomorphisms of an irrational 2-Torus

Let $T^2_{\alpha,\beta} = T_\alpha \times T_\beta$, where $T_\alpha = \mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z})$ and $T_\beta = \mathbf{R}/(\mathbf{Z} + \beta\mathbf{Z})$, with $\alpha, \beta \in \mathbf{R} - \mathbf{Q}$. Let $\omega = \theta_\alpha \wedge \theta_\beta$, where θ_α and θ_β are the projection of dt on T_α and T_β. That is, $\pi_\alpha^*(\theta_\alpha) = \pi_\beta^*(\theta_\beta) = dt$, where $\pi_\alpha : \mathbf{R} \to T_\alpha$ and $\pi_\beta : \mathbf{R} \to T_\beta$ are the projections.

317. The diffeomorphisms. Let $\Delta : \Phi \mapsto \big((\tau, \tau'), \phi\big)$ be the map, defined on $\mathrm{Diff}(T^2_{\alpha,\beta})$ by

$$(\tau, \tau') = \Phi(1, 1) \quad \text{and} \quad \phi(z, z') = (\bar\tau, \bar\tau') \cdot \Phi(z, z').$$

The pair (τ, τ') belongs to $T^2_{\alpha,\beta}$, the bar $\bar\tau$ denotes the inverse τ^{-1}, the dot $\cdot$ denotes the group multiplication on $T^2_{\alpha,\beta}$, and ϕ belongs to the subgroup $\mathrm{Diff}(T^2_{\alpha,\beta}, \mathbf{1})$ of diffeomorphisms fixing $\mathbf{1} = (1, 1)$.

(1) The map Δ is a diffeomorphism from $\mathrm{Diff}(T^2_{\alpha,\beta})$ to the product $T^2_{\alpha,\beta} \times \mathrm{Diff}(T^2_{\alpha,\beta}, 1)$, where $T^2_{\alpha,\beta}$ is equipped with its diffeology and $\mathrm{Diff}(T^2_{\alpha,\beta}, 1)$ with the functional diffeology.
(2) Acting by multiplication, the group $T^2_{\alpha,\beta}$ is naturally contained in $\mathrm{Diff}(T^2_{\alpha,\beta})$. It is its neutral component:
$$\mathrm{Diff}(T^2_{\alpha,\beta}, 1)^\circ = T^2_{\alpha,\beta}.$$
(3) The subgroup $\mathrm{Diff}(T^2_{\alpha,\beta}, 1)$ is discrete and identifies with the group of components of $\mathrm{Diff}(T^2_{\alpha,\beta})$:
$$\pi_0(\mathrm{Diff}(T^2_{\alpha,\beta})) \simeq \mathrm{Diff}(T^2_{\alpha,\beta}, 1).$$

Proof. Since $T^2_{\alpha,\beta}$ is a diffeological group (multiplication and inversion smooth), the map Δ is smooth. Its inverse is given by
$$\Delta^{-1} : \big((\tau, \tau'), \phi\big) \mapsto [\Phi : (z, z') \mapsto (\tau, \tau') \cdot \phi(z, z')].$$
And for the same reasons, Δ^{-1} is smooth too. Therefore, Δ is a diffeomorphism.

Now, let us prove that $\mathrm{Diff}(T^2_{\alpha,\beta}, 1)$ is discrete, for the functional diffeology. Let $\phi \in \mathrm{Diff}(T^2_{\alpha,\beta}, 1)$, and consider the composite $\phi \circ \pi : \mathbf{R}^2 \to T^2_{\alpha,\beta}$, where $\pi : \mathbf{R}^2 \to T^2_{\alpha,\beta}$ is the projection. Since $\mathbf{R}^2$ is simply connected, and since $\phi(1, 1) = (1, 1)$, there exists a unique smooth map $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$ such that $\Phi(0, 0) = (0, 0)$ and $\pi \circ \Phi = \phi \circ \pi$.

$$\begin{array}{ccc} \mathbf{R}^2 & \xrightarrow{\ \Phi\ } & \mathbf{R}^2 \\ \pi \downarrow & & \downarrow \pi \\ T^2_{\alpha,\beta} & \xrightarrow[\ \phi\ ]{} & T^2_{\alpha,\beta} \end{array}$$

Now, let us denote by K the group $(\mathbf{Z} + \alpha\mathbf{Z}) \times (\mathbf{Z} + \beta\mathbf{Z}) \subset \mathbf{R}^2$. The group $T^2_{\alpha,\beta}$ is just the quotient $\mathbf{R}^2/K$. The map Φ satisfies then
$$\Phi(X + K) = \Psi(X) + K'.$$
Where K' depends *a priori* on X and K. But since $X \mapsto \Phi(X+K) - \Phi(X)$ is smooth, and since $K \subset \mathbf{R}^2$ is discrete, K' depends only on K. Thus,

the tangent linear map satisfies

$$D(\Phi)(X+K) = D(\Phi)(X),$$

for all X and all K. Choosing $X = 0$, we have, for all $K \in \mathcal{K}$, $D(\Phi)(K) = M$, with $M = D(\Phi)(0) \in L(\mathbf{R}^2)$. But since $X \mapsto D(\Phi)(X)$ is smooth and $\mathcal{K}$ is dense in $\mathbf{R}^2$, $D(\Phi)(X) = M$ for all X. But that gives the expression of Φ, which is a linear map:

$$\Phi(X) = MX \quad \text{with} \quad M \in GL(\mathbf{R}^2).$$

Remark that *a priori* M is just linear, but if it has a non-zero kernel then its image by π in $T^2_{\alpha,\beta}$ is (at least) a whole 1-dimensional subspace mapped into $(1,1)$ by ϕ. But ϕ is a diffeomorphism and that cannot happen. Therefore, M is non degenerate, and $\mathrm{Diff}(T^2_{\alpha,\beta}, 1)^\circ$ is naturally identified to a subgroup of $GL(\mathbf{R}^2)$. Now, since $\pi \circ \Phi = \phi \circ \pi$ and $\phi(1) = 1$, $M\mathcal{K} \subset \mathcal{K}$. Actually, since M is the lifting on the universal covering of a diffeomorphism, $M\mathcal{K} = \mathcal{K}$. Thus, for all n, m, n', m' in $\mathbf{Z}$, there exist n'', m'', n''', m''' in $\mathbf{Z}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n+\alpha m \\ n'+\beta m' \end{pmatrix} = \begin{pmatrix} n''+\alpha m'' \\ n'''+\beta m''' \end{pmatrix} \quad \text{with} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We shall not solve this equation because it is not the purpose of this exercise, but let us just notice that:

$$\begin{matrix} a, b \in \mathbf{Z}+\alpha\mathbf{Z} \\ c, d \in \mathbf{Z}+\beta\mathbf{Z} \end{matrix} \quad \& \quad \begin{matrix} \alpha a, \beta b \in \mathbf{Z}+\alpha\mathbf{Z} \\ \alpha c, \beta d \in \mathbf{Z}+\beta\mathbf{Z} \end{matrix}.$$

From these conditions, we get immediately that the subgroup in $GL(\mathbf{R}^2)$ identified to $\mathrm{Diff}(T^2_{\alpha,\beta}, 1)$ is discrete (diffeologically). Hence, since $\mathrm{Diff}(T^2_{\alpha,\beta}) \simeq T^2_{\alpha,\beta} \times \mathrm{Diff}(T^2_{\alpha,\beta}, 1)$, where $T^2_{\alpha,\beta}$ is connected and $\mathrm{Diff}(T^2_{\alpha,\beta}, 1)$ is discrete, the group $\mathrm{Diff}(T^2_{\alpha,\beta}, 1)$, identifies naturally with $\pi_0(\mathrm{Diff}(T^2_{\alpha,\beta}))$. ▶

87. The moment map of the torus action onto itself

We consider now the action of $\mathbf{R}^2$ on $T^2_{\alpha,\beta}$ defined by

$$(a,b)_{T^2_{\alpha,\beta}}(z,z') = \pi \circ (a,b)_{\mathbf{R}^2}(x,x') = (\pi_\alpha(x+a), \pi_\beta(x'+b)),$$

with

$$(z, z') = \pi(x, x'), \text{ that is, } z = \pi_\alpha(x) \text{ and } z' = \pi_\beta(x').$$

The paths moment map of this action of $\mathbf{R}^2$ on $(T^2_{\alpha,\beta}, \omega)$ is defined by

$$\Psi(\gamma) = \hat{\gamma}^*(\mathcal{K}\omega),$$

where $\mathcal{K}$ is the Chain-Homotopy Operator, $\gamma \in \text{Paths}(T^2_{\alpha,\beta})$ and $\hat{\gamma}$ is the orbit map $\hat{\gamma} : \mathbf{R}^2 \to \text{Paths}(T^2_{\alpha,\beta})$, defined by

$$\hat{\gamma} : (a, b) \mapsto [t \mapsto (a, b)_{T^2_{\alpha,\beta}} \gamma(t)].$$

Let $\gamma : t \mapsto (z(t), z'(t))$ be a path in $T^2_{\alpha,\beta}$. Thanks to the Monodromy Theorem, there always exists a lift $\bar{\gamma} : t \mapsto (x(t), x'(t))$ of γ into $\mathbf{R}^2$,

$$\gamma(t) = \pi(\bar{\gamma}(t)) = (\pi_\alpha(x(t)), \pi_\beta(x'(t))).$$

It is uniquely defined by γ and the image of 0. We denote by

$$\pi_* : \text{Paths}(\mathbf{R}^2) \to \text{Paths}(T^2_{\alpha,\beta}), \text{ with } \pi_*(\bar{\gamma}) = \pi \circ \bar{\gamma},$$

the projection defined by the composite.

318. Exercise. (The moment map) Let γ be a path in $T^2_{\alpha,\beta}$ and $\bar{\gamma}$ be a lift of γ into $\mathbf{R}^2$. Show that

$$\Psi(\gamma) = \Psi(\bar{\gamma}),$$

where Ψ denotes the paths moment map of $\mathbf{R}^2$ acting on $T^2_{\alpha,\beta}$ for the left-hand side, and acting on $\mathbf{R}^2$ on the right-hand side.

Proof. We have:

$$\begin{aligned}
\hat{\gamma}(a, b) &= [t \mapsto (a, b)_{T^2_{\alpha,\beta}}(z(t), z'(t))] \\
&= [t \mapsto (\pi_\alpha(x(t) + a), \pi_\alpha(x'(t) + b))] \\
&= [t \mapsto \pi(x(t) + a, x'(t) + b)] \\
&= \pi \circ (a, b)_{\mathbf{R}^2} \circ \bar{\gamma} \\
&= \pi_*(\hat{\bar{\gamma}}(a, b)) \\
&= \pi_* \circ \hat{\bar{\gamma}}(a, b),
\end{aligned}$$

and then,

$$\hat{\gamma} = \pi_* \circ \hat{\text{ɣ}}.$$

And we note that that apply to any lift ɣ of γ. Injecting that into the expression of the paths moment map above, we get

$$\Psi(\gamma) = \widehat{\pi_* \circ \hat{\text{ɣ}}}^{*}(\mathcal{K}\omega) = \hat{\text{ɣ}}^*\big((\pi_*)^*(\mathcal{K}\omega)\big).$$

But, according to the variance of the paths moment map,

$$(\pi_*)^* \circ \mathcal{K} = \mathcal{K} \circ \pi^*.$$

Thus

$$\Psi(\gamma) = \hat{\text{ɣ}}^*(\mathcal{K}(\pi^*(\omega))) = \hat{\text{ɣ}}^*(\mathcal{K}(dx \wedge dy)) = \Psi(\text{ɣ}),$$

And that is what had to be proven. ▶

319. Exercise: The universal path moment map. Let γ be a path in $T^2_{\alpha,\beta}$ and ɣ be a lift of γ in $\mathbf{R}^2$. Let ɣ$(0) = (a, b)$ and ɣ$(1) = (a', b')$. Show that the path moment map relatively to $\mathbf{R}^2$ is given by

$$\Psi(\gamma) = (a' - a)dy - (b' - b)dx.$$

And that the universal moment map is then

$$\Psi_\omega(\gamma) = (a' - a)\theta_\beta - (b' - b)\theta_\alpha.$$

☞ Proof. According to the previous exercise, $\Psi(\gamma) = \Psi(\text{ɣ})$. Next, since $\Psi(\text{ɣ})$ is fixed-ends homotopic invariant, we can always choose ɣ to be the affine path connecting its origin to its end:

$$\Psi(\text{ɣ}) = \Psi([t \mapsto t(a', b') + (1 - t)(a, b)]).$$

The expression of the moment map in this case is simlpy given by

$$\Psi(\text{ɣ})(\mathrm{P})_r(\delta r) = \int_0^1 (dx \wedge dy)_{\text{ɣ}(t)}(\dot{\text{ɣ}}(t), \delta\text{ɣ}(t))\ dt,$$

with

$$\dot{\text{ɣ}}(t) = \begin{pmatrix} a' - a \\ b' - b \end{pmatrix} \quad \& \quad \delta\text{ɣ}(t) = [\mathrm{D}(\mathrm{P}(r))(\text{ɣ}(t))]^{-1} \frac{\partial \mathrm{P}(r)(\text{ɣ}(t))}{\partial r}(\delta r),$$

where P is a plot of $\mathbf{R}^2$. Now, the moment map $\Psi(\text{ɣ})$ is an invariant 1-form on $\mathbf{R}^2$, thus $\Psi(\text{ɣ}) = \mathrm{A}dx + \mathrm{B}dy$. The coefficients A and B are given by $\Psi(\gamma)[s \mapsto T_{(s,0)}]_s(1)$ and $\Psi(\gamma)[s \mapsto T_{(0,s)}]_s(1)$, where $T_{(x,y)}$ is the translation of (x, y). In our case $[D(P(r))(\gamma(t))] = \mathbf{1}$,

$$\frac{\partial T_{(s,0)}(x,y)}{\partial s}(1) = (1,0) \text{ and } \frac{\partial T_{(0,s)}(x,y)}{\partial s}(1) = (0,1).$$

Hence,

$$A = \int_0^1 (dx \wedge dy) \begin{pmatrix} a'-a \\ b'-b \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt = -(b'-b)$$

and

$$B = \int_0^1 (dx \wedge dy) \begin{pmatrix} a'-a \\ b'-b \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt = a'-a.$$

Thus, $\Psi(\gamma) = (a'-a)dy - (b'-b)dx$. Now, since $\mathrm{Diff}(T^2_{\alpha,\beta})^\circ = T^2_{\alpha,\beta}$, and $T^2_{\alpha,\beta} = \mathbf{R}^2/(\mathbf{Z}+\alpha\mathbf{Z}) \times (\mathbf{Z}+\beta\mathbf{Z})$, Ψ_ω is the pushforward of Ψ, that is, $\Psi_\omega(\gamma) = (a'-a)\theta_\beta - (b'-b)\theta_\alpha$. ▶

320. Exercise: The holonomy. Let Γ_ω be the universal holonomy of ω. We know that the neutral component of $\mathrm{Diff}(T^2_{\alpha,\beta}, \omega)$ is isomorphic to $T^2_{\alpha,\beta}$ acting on itself by multiplication. Show that

$$\Gamma_\omega = \{k\theta_\alpha + k'\theta_\beta \mid k \in \mathbf{Z}+\beta\mathbf{Z} \text{ and } k' \in \mathbf{Z}+\alpha\mathbf{Z}\}.$$

Proof. By definition

$$\Gamma_\omega = \{\Psi_\omega(\ell) \mid \ell \in \mathrm{Loops}(T^2_{\alpha,\beta})\}.$$

We can choose the loops ℓ pointed at the origin. Every loop pointed at the origin is the projection on a path $\gamma = [t \mapsto t(k,k')]$, where $k \in \mathbf{Z}+\alpha\mathbf{Z}$ and $k' \in \mathbf{Z}+\beta\mathbf{Z}$. Now, according to the previous exercise we get immediately $\Psi_\omega(\gamma) = k\theta_\beta - k'\theta_\alpha$. ▶

88. The symplectic irrational torus

We already proposed, in a few papers, a definition of a *symplectic diffeological space*. The objects and constructions used here can be found in [PIZ10]

321. Definition. *A closed 2-form ω defined on a diffeological space X is* symplectic *if it satisfies the following two conditions:*

(1) *The space* X *is locally homogenous under the action of local automorphisms. Precisely, that means that for every pair of points* x, x' *in* X*, there exists a local diffeomorphism* ϕ *of* X *such that:*

$$\phi^*(\omega) = \omega \upharpoonright \mathrm{dom}(\phi) \quad \textit{and} \quad \phi(x) = x'.$$

(2) *The universal moment map* $\mu_\omega : \mathrm{X} \to \mathcal{G}^*_\omega/\Gamma_\omega$ *is injective.*[1]

The first condition[2] describes a *presymplectic space*. The second condition mimic what happens for symplectic manifolds. This definition could be possibly modified if the development of "Symplectic Diffeology" requires it, but it captures already quite well the spirit of what we regard as symplectic.

Note however that the second condition has a better formulation using the 2-points moment map ψ_ω, with values in $\mathcal{G}^*_\omega/\Gamma_\omega$, that is,

(2') The diagonal $\Delta\mathrm{X} = \{(x, x)\}_{x\in\mathrm{X}}$ is the zero locus of the 2-points moment map ψ.

This condition has an expression involving only the paths moment map:

(2") For all $\gamma \in \mathrm{Paths(X)}$, $\Psi(\gamma) \in \Gamma_\omega$ if and only if $\gamma \in \mathrm{Loops(X)}$.

322. Exercise: Symplectic irrational 2-torus. Show that the irrational 2-torus $\mathrm{T}^2_{\alpha,\beta}$, equipped with the 2-form ω, is symplectic.

☞ Proof. We know already that the neutral component of $\mathrm{Diff}(\mathrm{T}^2_{\alpha,\beta}, \omega)$ is $\mathrm{T}^2_{\alpha,\beta}$ itself. Then $\mathrm{Diff}(\mathrm{T}^2_{\alpha,\beta}, \omega)$ is transitive and ω is presymplectic. Now, the paths moment map for a path γ connecting $z = \pi(a, b)$ to $z' = \pi(a', b')$ is given by

$$\Psi_\omega(\gamma) = (a' - a)\theta_\beta - (b' - b)\theta_\alpha.$$

Then, $\Psi_\omega(\gamma) \in \Gamma_\omega$ if and only if $a' - a \in \mathbf{Z} + \alpha\mathbf{Z}$ and $b' - b \in \mathbf{Z} + \beta\mathbf{Z}$, that is, $a' = a + k$ and $b' = b + k'$, with $k \in \mathbf{Z} + \alpha\mathbf{Z}$ and $k' \in \mathbf{Z} + \beta\mathbf{Z}$. Thus, $z' = \pi(a + k, b + k') = \pi(a, b) = z$ and γ is a loop, and the

[1] It is safer to ask for a covering onto its image.

[2] There is an alternative between local transitivity or local homogeneity actually.

condition (2”) is satisfied. Since the two condition (1) and (2”) are satisfied, $(\mathrm{T}^2_{\alpha,\beta}, \omega)$ is a symplectic diffeological space. ▶

There are two equivalent ways of considering Hamiltonian diffeomorphisms, described in [**TB**, § 9.15] and [**TB**, § 9.16]. Use both of them in the next exercise.

323. Exercise: Hamiltonian diffeomorphisms. Show that the group of Hamiltonian diffeomorphisms of $(\mathrm{T}^2_{\alpha,\beta}, \omega)$ is trivial.

☞ Proof. Considering the definition [**TB**, § 9.15], we first construct the group $\widehat{\mathcal{H}}_\omega$ as the intersection of the morphisms f_ε, from the covering of the neutral component of $\mathrm{Diff}(\mathrm{T}^2_{\alpha,\beta}, \omega)$ to $\mathbf{R}$, integrating the closed 1-forms $\varepsilon \in \Gamma_\omega$. And then, the group $\mathrm{Ham}(\mathrm{T}^2_{\alpha,\beta}, \omega)$ is the projection of $\widehat{\mathcal{H}}_\omega$ in $\mathrm{Diff}(\mathrm{T}^2_{\alpha,\beta}, \omega)$. In our case ε write $k\theta_\alpha + h\theta_\beta$, where $k \in \mathbf{Z} + \beta\mathbf{Z}$ and $h \in \mathbf{Z} + \alpha\mathbf{Z}$. Its integration function is $f_\varepsilon : (x, y) \mapsto kx + hy$. Its kernel is the line $\ker(\varepsilon) = \{t(-h, k)\}_{t\in\mathbf{R}}$. Therefore the intersection $\widehat{\mathcal{H}}_\omega$ is reduced to $\{(0, 0)\}$, and $\mathrm{Ham}(\mathrm{T}^2_{\alpha,\beta}, \omega) = \{\mathbf{1}\}$.

The second way describes the element of $\mathrm{Ham}(\mathrm{T}^2_{\alpha,\beta}, \omega)$ as the end of paths $t \mapsto f_t$ in $\mathrm{Diff}(\mathrm{T}^2_{\alpha,\beta}, \omega)$, centered at the identity, for which there exists a path $t \mapsto \Phi_t$ in $\mathcal{C}^\infty(\mathrm{T}^2_{\alpha,\beta}, \mathbf{R})$ such that $i_{\mathrm{F}_t}(\omega) = -d\Phi_t$, where $\mathrm{F}_t = [s \mapsto f_t^{-1} \circ f_{t+s}]$ [**TB**, § 9.16]. But in our case, $\mathcal{C}^\infty(\mathrm{T}^2_{\alpha,\beta}, \mathbf{R})$ is reduced to the constant maps, then for all t, $d\Phi_t = 0$, that is, $i_{\mathrm{F}_t}(\omega) = 0$. Let $p \in \mathrm{Paths}(\mathrm{T}^2_{\alpha,\beta})$ and let $[t \mapsto f_t]$ be a path in $\mathrm{Ham}(\mathrm{T}^2_{\alpha,\beta}, \omega) \subset \mathrm{Diff}(\mathrm{T}^2_{\alpha,\beta}, \omega)$. Thanks to formula [**TB**, § 9.2 (♣)], we have

$$\Psi_\omega(p)([t \mapsto f_t])_t(1) = -\int_p i_{\mathrm{F}_t}(\omega) = 0,$$

for all path $[t \mapsto f_t]$ in $\mathrm{Ham}(\mathrm{T}^2_{\alpha,\beta}, \omega)$. Now, on the one hand, $\Psi_\omega(p) = (a' - a)\theta_\beta - (b' - b)\theta_\alpha$, where $p(0) = \pi(a, b)$ and $p(1) = \pi(a', b')$. On the other hand, $t \mapsto f_t$ has a lifting in $\mathbf{R}^2$, $f_t = (\pi_\alpha(x(t)), \pi_\beta(y(t)))$. Then, $\Psi_\omega(p)([t \mapsto f_t])_t(1) = (a' - a)\dot{y}(t) - (b' - b)\dot{x}(t) = 0$, for all $a' - a$ and $b' - b$. Thus, $\dot{x}(t) = 0$ and $\dot{y}(t) = 0$. Therefore $f_t = \mathbf{1}$ for all t, and $\mathrm{Ham}(\mathrm{T}^2_{\alpha,\beta}, \omega) = \{\mathbf{1}\}$. ▶

The Diffeology Framework of General Covariance

In this lecture[1] I will present the principle of general covariance, introduced by J.-M. Souriau in the 70s, as the active version of the principle of general relativity. I will show how we can include it in the formal diffeology framework.

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89. The principle of general covariance

Jean-Marie Souriau established its *principle of general covariance* in a few papers. Namely, the first two as short notes [**Sou70a, Sou70b**],

- *Sur le mouvement des particules à spin en relativité générale*. C. R. Acad. Sci. Paris Sér. A, 271 :751–753, 1970.

- *Sur le mouvement des particules dans le champs életromagnétique*. C. R. Acad. Sci. Paris Sér. A, 271 :1086–1088, 1970.

Profusely deveoped and completed in a famous paper [**Sou74**]

- *Modèle de particule à spin dans le champ électromagnétique et gravitationnel*. Ann. Inst. Henri Poincaré, XX A, 1974.

The purpose of these papers was to describe the motion of a material medium submitted to the action of a gravitational field, represented by a pseudo-Riemmannian metric g, of signature $(+---)$, on a manifold M with dimension 4, the *space-time*.

[1] Talk given at the "Theory of Gravitation and Variation in Cosmology" School, CIRM/Luminy, April 12–16, 2021.

Of course that depends on the nature of the medium: if it is a continuous medium or a system of particles, in presence of electromagnetic field or not, with or without spin...

The motion, the evolution, of these system are described by differential equations, ordinary or partial differential systems. In his papers, Souriau proposes a unique process, a mechanism, which gives them all at the same time, according to the same principle of general covariance, which can be regarded as the effective *principle of general relativity*.

I should mention that the principle of general covariance is not the only way to find the equations of motion of material medium in general relativity. There is obviously the classical variational approach, the motion are the extremals of some *action functions* defined by a Lagrangian. I will show later how the two approaches are related.

The point is that it is not apparent if and how the variational approach expresses the principle of general relativity, but there is no doubt about the principle of general covariance. We'll discuss it.

324. The Physis. As we said above, let M be a manifold of dimension 4. To describe the motion of a passive material in the gravitational field we shall consider the infinite dimensional space of all the pseudo-Riemmannian metric on M. Let $\mathcal{M}$ be this space.

$$\mathcal{M} = \{g : \text{pseudo-Riemannian metric} \mid \text{sgn}(g) = (+---)\}$$

Now consider the group Diff(M) of diffeomorphism on M, acting by pullback (or pushforward) on M: For all $f \in \text{Diff}(\text{M})$, for all $g \in \mathcal{M}$,

$$f^*(g)_x(v, w) = g_{f(x)}\big(\text{D}f_x(v), \text{D}f_x(w)\big).$$

Now, the indifference of the real world with respect to its different representations leads to regard the gravitational field equivalently represented by g or any other metric g' on the same orbit of the group Diff(M). In other words, the gravitational field is not g but the orbit

$$\text{class}(g) = \text{Diff}(\text{M})^*(g).$$

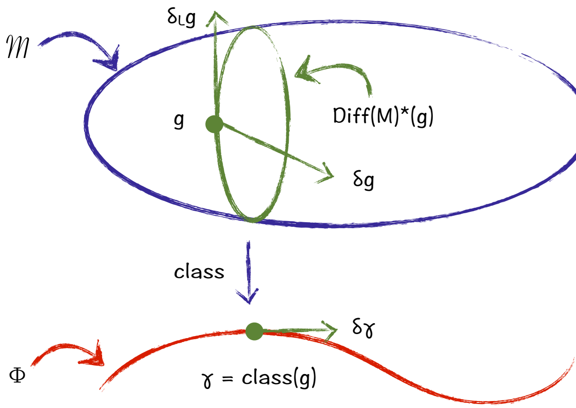


Figure 44. The Physis.

What Souriau proposes is to regard the real world, in this case: the gravitation, represented not by $\mathcal{M}$ but by the quotient

$$\Phi = \mathcal{M}/\mathrm{Diff}(\mathrm{M});$$

Originally, Souriau called the quotient $\mathcal{M}/\mathrm{Diff}(\mathrm{M})$ the *Hyperspace* and denoted it by $\mathcal{H}$, but in his last unpublished works ("La Grammaire de le Nature") he changed for *Physis*, since for him the nature expresses itself through this quotient. We decide to follow him and denote it by the greek letter Φ.

325. The passive matter. There are two kinds of evolution of matter in relativity: passive and field generating. The field generating matter is described by Einstein's equations. We may come back to that point later, but now we are interested first in the *Passive Equations of Physics*, as named by Shlomo Sternberg in [**Ste12**]:

- *General Covariance and the Passive Equations of Physics.*

They are defined by the following principle:

Souriau's principle of general covariance *The evolution of matter submitted to a gravitational field is described by a* covector *on the space* $\Phi = \mathcal{M}/\mathrm{Diff}(\mathrm{M})$.

Covectors on the space Φ are defined by a universal equation we shall establish now.

326. The universal equation of passive matter. We consider a metric

$$g \in \mathcal{M} \quad \text{and} \quad \gamma = \text{class}(g) \in \Phi.$$

Let us interpret a covector at the point γ as a linear functional defined on the variation $\delta\gamma$. The point is to understand what is $\delta\gamma$, *a priori* it would be a "tangent vector" at the point γ. That is, a tangent vector δg at the point g, modulo a "vertical vector", generated by the infinitesimal action of the group Diff(M).

Thus, a linear functional defined on the variation $\delta\gamma$ can be interpreted as a linear functional, defined on the variations δg but vanishing on the vertical vectors.

The variation δg are naturally elements of the vector space of covariant symmetric 2-tensors that we denote by $\mathcal{E}$, $\delta g \in \mathcal{E}$.

The vertical vectors are generated by derivation of paths along the orbits of Diff(M), that is,

$$\left.\frac{\partial f_s^*(g)}{\partial s}\right|_{s=0}, \text{ with } [s \mapsto f_s] \in \text{Paths}(\text{Diff}(\text{M})), \text{ and } f_0 = \mathbf{1}_{\text{M}}.$$

Here, Paths(Diff(M)) denote the smooth paths in Diff(M) in the sense that $(s,x) \mapsto f_s(x)$ is smooth, the variable s being defined on some open neighbourhood of $0 \in \mathbf{R}$.

Then, let us define the vector field ξ on M by

$$\xi(x) = \left.\frac{\partial f_s(x)}{\partial s}\right|_{s=0}$$

Thus, the vertical vectors write:

$$\left.\frac{\partial f_s^*(g)}{\partial s}\right|_{s=0} = \left.\frac{\partial \exp(s\xi)^*(g)}{\partial s}\right|_{s=0}$$

That is, the Lie derivative of the metric g by the vector field ξ, we denote it by

$$\delta_{\text{L}} g, \text{ or } \pounds_\xi(g), \quad \text{with} \quad \delta x = \xi(x)$$

In terms of coordinates, the Lie derivative writes:

$$\big(\delta_L(g)\big)_{\mu\nu} = \hat{\partial}_\mu \xi_\nu + \hat{\partial}_\nu \xi_\mu,$$

where $\hat{\partial}$ denotes the covariant derivative with respect to the metric g. The symbol $\hat{\partial}$ is also denoted by ∇, but I prefer the first notation because everything with a hat will refer to a covariant differentiation.

In conclusion, a covector at the point $\gamma = \mathrm{class}(g)$ will be represented by a linear fuctional, let us say $\mathcal{T}$, defined on the space $\mathcal{E}$ of covariant symmetric 2-tensors δg, and vanishing on the Lie derivative of the metric by any vector field on M. We can write

$$T^*_\gamma \Phi = \{\mathcal{T} \in \mathcal{E}^* \mid \mathcal{T}(\delta_L g) = 0 \text{ for all } \delta x \in \mathrm{Vect}(M)\},$$

where $\mathcal{E}^*$ is the dual in some sense we do not specify now, and Vect(M) is the space of the vector fields on M.

In other words, the passive motion of the matter in a gravitational field g is described by a functional $\mathcal{T} \in \mathcal{E}^*$ satisfying the universal equation:

$$\mathcal{T}(\delta_L g) = 0 \text{ for all } \delta x \in \mathrm{Vect}(M). \qquad (\clubsuit)$$

Such functionals $\mathcal{T}$ are called *Eulerian functionals*, and ($\clubsuit$) can be called the *Euler-Souriau equation*. We shall understand later this choice of vocabulary.

Important Note. Actually, for technical reasons and maybe more deeper reasons, all the variations considered here, all vector fields (which are also variations) will be supposed *compactly supported*. We will denoted these spaces with a subscript K, for example $\mathrm{Vect}_K(M)$, $\mathrm{Diff}_K(M)$ etc.

327. The continuous medium. In this framework, continuous medium are described by a continuous distribution of a contravariant symmetric 2-tensor T, such that

$$\mathcal{T}(\delta g) = \frac{1}{2} \int_M T^{\mu\nu} \delta g_{\mu\nu} \,\mathrm{vol},$$

where vol is the Riemmannian volume associated with g.

Let us write that $\mathcal{T}$ is Eulerian:

$$0 = \mathcal{T}(\delta_L g) = \frac{1}{2}\int_M T^{\mu\nu}\delta_L g_{\mu\nu}\,\mathrm{vol}$$
$$= \int_M T^{\mu\nu}\hat{\partial}_\mu \xi_\nu\,\mathrm{vol} = \int_M \hat{\partial}_\mu(T^{\mu\nu}\xi_\nu)\,\mathrm{vol} - \int_M \left(\hat{\partial}_\mu T^{\mu\nu}\right)\xi_\nu\,\mathrm{vol}\,.$$

The first term is zero because

$$\hat{\partial}_\mu(T^{\mu\nu}\xi_\nu) = \hat{\mathrm{d}}\mathrm{iv}(\eta), \quad \text{with} \quad \eta^\mu = \sum_\mu T^{\mu\nu}\xi_\nu,$$

and

$$\int_M \hat{\mathrm{d}}\mathrm{iv}(\eta)\,\mathrm{vol} = \int_M \delta_L(\mathrm{vol}) = \int_M d[\mathrm{vol}(\eta)] = \int_{\partial M} \mathrm{vol}(\eta) = 0,$$

since η is compactly supported.

Thus, it remains from $\mathcal{T}(\delta_L g) = 0$, the equation:

$$\int_M \left(\hat{\partial}_\mu T^{\mu\nu}\right)\xi_\nu\,\mathrm{vol} = 0,$$

for all compactly supported vector field ξ. That is equivalent to:

$$\hat{\mathrm{d}}\mathrm{iv}(T) = 0,$$

which is the conservation equations of a continuous medium in general relativity, also called *Euler equations*. And that explains the choice for the name “Eulerian distribution” of the distributions $\mathcal{T}$ satisfying the condition (♣).

328. The geodesics. Now, we look for an Eulerian distribution supported by a curve, let us say $t \mapsto x$. Thus, for all δg,

$$\mathcal{T}(\delta g) = \frac{1}{2}\int_{-\infty}^{+\infty} T^{\mu\nu}\delta g_{\mu\nu}\,dt.$$

The Eulerian condition writes:

$$0 \;=\; \int_{-\infty}^{+\infty} T^{\mu\nu}\hat{\partial}_\mu \xi_\nu\,dt,$$

for all compactly supported vector field ξ. For the sake of simplicity we assume that the curve exits every compact.

Let us multiply the vector field ξ by a real function α that vanishes on the curve, we get another compactly supported vector field, but we get now:

$$0 = \int_{-\infty}^{+\infty} T^{\mu\nu}\hat{\partial}_\mu(\alpha\xi_\nu)\,dt = \int_{-\infty}^{+\infty} T^{\mu\nu}\partial_\mu(\alpha)\xi_\nu\,dt + \int_{-\infty}^{+\infty} \alpha T^{\mu\nu}\hat{\partial}_\mu\xi_\nu\,dt$$
$$= \int_{-\infty}^{+\infty} T^{\mu\nu}N_\mu\xi_\nu\,dt + 0 \quad \text{(since } \alpha \text{ vanishes on the curve)},$$

with

$$N = \mathrm{grad}(\alpha) = g^{-1}\left(\frac{\partial\alpha}{\partial x}\right)$$

Thus,

$$T^{\mu\nu}N_\mu = 0$$

for all vector N orthogonal to the curve. Hence, there exists a vector P such that

$$T^{\mu\nu} = P^\mu\frac{dx^\nu}{dt},$$

and since $T^{\mu\nu}$ is symmetric, the vector P is parallel to the curve:

$$P \propto \frac{dx}{dt}.$$

Now, we put that in the universal equation, that gives:

$$0 = \int_{-\infty}^{+\infty} P^\mu\frac{dx^\nu}{dt}\hat{\partial}_\nu\xi_\mu\,dt = \int_{-\infty}^{+\infty} P^\mu\frac{\hat{\partial}\xi_\mu}{\partial x_\nu}\frac{dx^\nu}{dt}\,dt$$
$$= \int_{-\infty}^{+\infty} P^\mu\frac{\hat{d}\xi_\mu}{dt}\,dt = \int_{-\infty}^{+\infty}\frac{d}{dt}(P^\mu\xi_\mu)\,dt - \int_{-\infty}^{+\infty}\frac{\hat{d}P^\mu}{dt}\xi_\mu\,dt.$$

The first term vanishes since ξ is compactly supported and we have assumed that the curve leaves every compact. It remains:

$$\int_{-\infty}^{+\infty}\frac{\hat{d}P^\mu}{dt}\xi_\mu\,dt = 0$$

for all ξ. Therefore,

$$\frac{\hat{d}P}{dt} = 0.$$

The curve is geodesic and the vector P is of constant norm:

$$g(P,P) = P^\mu P_\mu = \mathrm{cst}$$

The value of the constant indicates the kind of geodesic of the curve supporting the distribution $\mathcal{T}$.

So, we see that the same condition, to be Eulerian emcompass at the same time the continuous media and also the geodesics.

329. Others... I will not develop more the examples in this short note, you can see [Sou74], and also [Ste12].

Let us say that the universal equation above (♣) gives all kind of passive equations of motion, of almost everything we can think about: string, charged particle in the elctromagnitic field (we should add the elecromagnetic potential A to the metric and the gauge transformation group), charged particle with spin in the gravitational and electromagnetic field and so on.

We can also find the laws or reflexion and diffraction of dust in a graviational field, along a discontinuity of the metric, by adapting the Eulerian condition to the group of diffeomorphism preserving the locus of the singularity [PIZ19].

330. Covariance and variation. As we said the calculus of variations is the classical way to get the equations of motions of passive mater in general relativity.

We have an action A depending on some fields σ in some given geometry g. This action is the integral of some Lagrangian function which depends on the specific problem. For example: for the geodesics σ is a curve and A is the lenght of the curve. And the equations of motions are the critical values of σ solutions of the equation:

$$\frac{\partial A}{\partial \sigma}(\delta\sigma) = 0.$$

Consider now the geometry as a part of the action A, that is, the action is a function of g and σ. Assume that there is a group **G** that acts on the space $\mathcal{M}$ of geometries and the space, and on the space Σ of fields σ such that:

(1) The action is invariant under the action of **G** acting diagonaly on $\mathcal{M} \times \Sigma$.

(2) The group $\mathbf{G}$ acts transitively on the space of fields.

Thus, the invariance of A will give the identity

$$\frac{\partial A}{\partial g}(\delta_L g) + \frac{\partial A}{\partial \sigma}(\delta_L \sigma) = 0,$$

where δ_L denotes a derivation along the 1-parameter subgroups of $\mathbf{G}$. Then, since G acts transitively on the space Σ, the identity above writes

$$\frac{\partial A}{\partial g}(\delta_L g) + \frac{\partial A}{\partial \sigma}(\delta \sigma) = 0,$$

for all $\delta\sigma$. Thus, if σ is a critical point of A for some value of the geometry g, then the covector

$$\mathcal{T} = \frac{\partial A}{\partial g} \quad \text{satisfies} \quad \mathcal{T}(\delta_L g) = 0,$$

for all $\delta\sigma$. The covector $\mathcal{T}$ is Eulerian.

Conversely, if the covector $\mathcal{T}$ is Eulerian for a value σ in some geometry g, then σ is a critical point of the action A.

The two principles give indeed the same solutions, they are miroring each other.

However, in absence of Lagrangian action, the principle of general covariance generalizes the *Principle of Least Action*.

It has the vertue, in addition to generalize the construction of the passive equations of physics, to *change the paradigm* of physics, from a metaphysical principle of least action to a geometrical invariance constraint, which is more palpable.

Remark. One should however note that the principle of general covariance does not find the irrational geodesics on the 2-torus, since the integral $\mathcal{T}(\delta g) = \frac{1}{2}\int_{-\infty}^{+\infty} P^{\mu} \frac{dx^{\nu}}{dt} \delta g_{\mu\nu} dt$ does not converge for all δg, except on closed geodesics.

331. The principle of general relativity. It is time to discuss the relationship between Einstein's *Principle of General Relativity* and Souriau's *principle of general covariance*. Usually the principle of general relativity is stated as follows:

- *physical laws are the same in all reference frames—inertial or non-inertial.*

This statement is vague enough to have made Vladimir Fock say: "there is no general relativity but a theory of gravitation". Indeed, what are the laws of physics, how do you express them? What does it mean to be the same in all frame? Same according to what. Yes, it is clearly unsatisfactory.

There is another formulation that we can find in the literature:

- *The equations of physics are covariant under the change of coordinates.*

Still unsatisfactory, too many terms are not weel defined.

On the other hand, the principle of general covariance is clear and unambiguous: the objects are well defined and the universal equation (♣) is unique and clear.

There is another difference between the usual statement of the principle of general relativity and the principle of general covariance:

The principle of general relativity is a passive way to refer to nature. It concerns the various means to describe parts of the world, through the various charts used to refer to the object.

The principle of general covariance is an active approach in the sense that the group of diffeomorphisms *acts* effectively on the geometries. The statement is not vague and says precisely: in general relativity, the objects of the nature are covectors of the Physis, quotient of the geometries by the group of diffeomorphisms of space-time.

90. The Diffeology framework

332. The Compact Diffeology on $\mathcal{M}$. Let $\mathcal{M}$ the set of pseudo-Riemmannian metrics on a manifold M, of signature $(+---)$. It is a subset of smooth maps from M to the covariant 2-tensors fiber-bundle $S_2(M)$ over M, they are smooth sections.

Since M and $S_2(M)$ are manifolds, $\mathcal{C}^\infty(M, S_2(M))$ is equipped with the functional diffeology, and $\mathcal{M} \subset \mathcal{C}^\infty(M, S_2(M))$ will be equipped with the subset diffeology.

Now, we will equip $\mathcal{M}$ with a sub-diffeology of the functional diffeology:

Definition. *A smooth parametretrization $r \mapsto g_r$ in $\mathcal{M}$, defined on some Euclidean domain* U, *is a plot of the* compact diffeology *if: for all $r \in U$ there is an open neighbourhood $V \subset U$ of r and a compact $K \subset M$ such that:*

$$\forall r' \in V, \forall x \in M - K, \quad g_{r'}(x) = g_r(x).$$

Indeed, the constant parametrizations satisfy that condition. The condition is obviously local. By composition with a smooth parametrization F in U, the neighbourhood V becomes $F^{-1}(U)$ for the same compact K.

Note. Let $r \mapsto g_r$ be a plot of the compact diffeology. Since M and $S_2(M)$ are manifolds the variation

$$\frac{\partial g_r(x)}{\partial r}(\delta r)$$

is a well defined symmetric 2-tensor on M, and is compactly supported. Indeed,

$$\frac{\partial g_r(x)}{\partial r}(\delta r) = \lim_{s \to 0} \frac{1}{s}\left[g_{r+s\delta r}(x) - g_r(x)\right].$$

For $\varepsilon \in \mathbf{R}$ sufficiently small, for all $|s| < \varepsilon$, $r + s\delta r$ will be contained in a ball contained in the neighbourhood V for which $g_{r'}(x) = g_r(x)$ outside some compact K. Thus, for $s \to 0$,

$$g_{r+s\delta r}(x) - g_r(x) = 0 \text{ outside } K,$$

and the variation of g_r is compactly supported.

333. Pushing forward pointed differential forms. In the chaper "On Riemannian metric in diffeology" we have seen the definition of pointed differential form, now, the general criterion of pushing forward differential forms can be adapted here. There are two cases, the general case of a subduction and the special case of *local subduction* [TB, § 2.16].

Let us recall that a subduction $\pi\colon Y \to X$ is said to be a local subduction if for all plots $P\colon U \to X$, for all $r \in U$ and all $y \in \pi^{-1}(x)$, $x = P(r)$, there exists a local lift Q in Y, that is, $\pi \circ Q = P \restriction \mathrm{dom}(Q)$, such that $Q(r) = y$.

In particular, projections on quotients by action of groups by diffeomorphisms are local subductions.

Let us come back to the general case. Let α_y a pointed k-form.

Proposition 1. *There exists a pointed k-form β_x, with $x = \pi(y)$, such that $\alpha_y = \pi^*(\beta_x)$ if and only if for all $y' \in \pi^{-1}(x)$ there is a pointed form $\alpha_{y'}$ satisfying: for all pair of plots P' and P'', pointed in the fiber $\pi^{-1}(x)$, such that $\pi \circ P' = \pi \circ P''$, $\alpha_{y'}(P') = \alpha_{y''}(P'')$ with $y' = P'(0)$ and $y'' = P''(0)$.*

If $X = Y/G$, where G is a diffeological group acting on Y by diffeomorphisms, the sitution is simpler because π is a local subduction. In this case, we have:

Proposition 2. *In that case $X = Y/G$, there exists a pointed k-form β_x, with $x = \pi(y)$, such that $\alpha_y = \pi^*(\beta_x)$ if and only if for all plots P' and P'' pointed in y, $\pi \circ P' = \pi \circ P''$ implies $\alpha_y(P') = \alpha_y(P'')$.*

334. Proposition: The Physis. Consider the group of diffeomorphisms with compact supports $\mathrm{Diff}_K(M)$, that is, the diffeomorphisms of M that are the identity outside of a compact.

The group $\mathrm{Diff}_K(M)$ acts on the set $\mathcal{M}$ of pseudo-Riemmannian metrics on M by pullback:

$$f^*\colon \mathcal{M} \to \mathcal{M}, \quad \text{with} \quad f^*(g)_x(v, w) = g_{f(x)}\big(\mathrm{D}f_x(v), \mathrm{D}f_x(w)\big).$$

Equip $\mathcal{M}$ with the compact diffeology. Let Φ be the quotient space:

$$\Phi = \mathcal{M}/\mathrm{Diff}_K(M)$$

Claim 1. *The group $\mathrm{Diff}_K(M)$ act on $\mathcal{M}$ by diffeomorphisms. Therefore, the projection* class$\colon \mathcal{M} \to \Phi$ *is a local subduction.*

Now, let us equip the group $\mathrm{Diff}_K(M)$ with the *compact functional diffeology*. A parametrization $P: r \mapsto f_r$, defined on U, in $\mathrm{Diff}_K(M)$ will be a plot if it is a plot for the functional diffeology and if:

Definition. *For all $r \in U$ there exists an open neigbourhood V of r and a compact $K \subset M$ such that: for all $r' \in V$, $f_{r'}$ is the identity outside K.*

Claim 2. *Equipped with the compact functional diffeology, the group $\mathrm{Diff}_K(M)$ acts smoothly on $\mathcal{M}$. That is, the map $f \mapsto f^*$ is smooth.*

Proof. First of all, f^* is bijective, its inverse is $(f^{-1})^*$. Now, let $P: r \mapsto g_r$ be a plot of the compact diffeology. Let $r \in \mathrm{dom}(P)$, there is an open neighbourhood V of r and a compact $K \in M$ such that: for all $r' \in V$, for all $x \in M - K$, $g_{r'}(x) = g_r(x)$. Thus, for all $r' \in V$, for all $x \in M - f^{-1}(K)$ $f^*(g_{r'})(x) = f^*(g_r)(x)$. The parametrization $f^* \circ P$ is then a plot of $\mathcal{M}$. The same holds for $(f^{-1})^*$. Therefore, f^* is a diffeomorphism on $\mathcal{M}$. Let us prove now that the action map:

$$\mathrm{Diff}_K(M) \times \mathcal{M} \to \mathcal{M} \quad \text{with} \quad (f, g) \mapsto f^*(g)$$

is smooth. Let $r \mapsto (f_r, g_r)$ be a plot, defined on some U. Let us show that $r \mapsto f_r^*(g_r)$ is smooth for the compact diffeology. It is already smooth for the functional diffeology, all that remains is the question of compact support. Let $r \in U$, there is an open neigbourhood V of r and two compacts $K, K' \subset M$ such that, for all $r' \in V$, for all $x \notin K$, $g_{r'}(x) = g_r(x)$ and for all $x \notin K'$, $f_{r'}(x) = f_r(x) = x$. We have to compare $f_r^*(g_r)(x)$ and $f_{r'}^*(g_{r'})(x)$. For all $r' \in V$ and $x \notin K'$, $f_{r'}(x) = f_r(x) = x$, then $f_{r'}^*(g_{r'})(x) = g_{r'}(x)$ and $f_r^*(g_r)(x) = g_r(x)$. Now, for all $x \notin K$, $g_{r'}(x) = g_r(x)$. Thus, if $r' \in V$ and $x \notin K'' = K \cup K'$, $f_r^*(g_r)(x) = f_{r'}^*(g_{r'})(x)$. Therefore, since K'' is a compact, $r \mapsto f_r^*(g_r)$ is a plot in $\mathcal{M}$ and the action is smooth. ▶

335. Pointed differential forms on the Physis. Let $\mathcal{T}$ be a real smooth bounded linear operator defined on the subspace $\mathcal{E} \subset S_2(M)$ of symmetric covariant 2-tensors with compact support. That is,

(1) $\mathcal{T}: \mathcal{E} \to \mathbf{R}$, and $\mathcal{T}(a\varepsilon + a'\varepsilon') = a\mathcal{T}(\varepsilon) + a'\mathcal{T}(\varepsilon')$, for all $a, a' \in \mathbf{R}$ and $\varepsilon, \varepsilon' \in \mathcal{E}$.

(2) $\mathcal{T} \in \mathcal{C}^\infty(\mathcal{E}, \mathbf{R})$.
(3) For all $\varepsilon \in \mathcal{E}$ there is a constant c such that: $|\mathcal{T}(\varepsilon)| \leq c\|\varepsilon\|$ for some norm on the space $\mathcal{E}$

Now, for all n-plots $r \to g_r$ pointed at $g = g_0$ let τ_g be defined by

$$\tau_g(r \mapsto g_r) \colon v \mapsto \mathcal{T}\left(\left.\frac{\partial g_r}{\partial r}\right|_{r=0} (v) \right) \quad \text{for all } v \in \mathbf{R}^n.$$

We know that since $r \mapsto g_r$ is a plot for the compact diffeology,

$$\frac{\partial g_r}{\partial r}(\delta r) \in \mathcal{E}.$$

Thus,

$$\mathcal{T}\left(\frac{\partial g_r}{\partial r}(\delta r)\right)$$

is well defined. So is $\tau_g(r \mapsto g_r)$, which belongs to $\Lambda^1(\mathbf{R}^n)$.

Claim 2. *The process τ_g is a differential 1-form pointed at g.*

Now,

Claim 3. *Consider two plots $\mathrm{P}\colon r \mapsto g_r$ and $\mathrm{P}\colon r \mapsto g'_r$ pointed at $g = g_0 = g'_0$. Assume* $\mathrm{class}(g_r) = \mathrm{class}(g'_r)$ *such that there exists a plot $r \mapsto f_r$ such that $g'_r = f_r^*(g_r)$, with $f_0 = 1_{\mathrm{M}}$. Then, $\tau_g(r \mapsto g_r) = \tau_g(r \mapsto g'_r)$ if and only if*

$$\mathcal{T}(\delta_{\mathrm{L}} g) = 0$$

for all compactly supported vector field δx.

In that case, there exists indeed a linear 1-form σ pointed at $\gamma =$ $\mathrm{class}(g)$ *such that $\tau_g =$* $\mathrm{class}^*(\sigma)$.

Thus, Eulerian distributions indeed represent special 1-form pointed somewhere in the quotient Φ.

Note. It remains to be seen what the obstacles are to the existence of such a smooth map $r \mapsto f_r$, when $\mathrm{class} \circ g'_r = \mathrm{class} \circ g_r$ in general. Probably, the subspace of pseudo-Riemannian metrics without isométries with compact support is a principal diffeological fiber bundle over it quotient by $\mathrm{Diff}_{\mathrm{K}}(\mathrm{M})$, and therefore such a smooth map $r \mapsto f_r$ always exists in this situation.

☞ Proof. Consider now τ_g, let $s \mapsto r \mapsto g_r$, where $s \mapsto r$ is a plot in $\mathbf{R}^n$ pointed at 0. Then,

$$\begin{aligned}\tau_g(s \mapsto g_r)(w) &= \mathcal{T}\left(\left.\frac{\partial g_r}{\partial s}\right|_{s=0}(w)\right) \\ &= \mathcal{T}\left(\left.\frac{\partial g_r}{\partial r}\right|_{r=0}\left(\left.\frac{\partial r}{\partial s}\right|_{s=0}(w)\right)\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}[s \mapsto r]^*\big(\tau_g(r \mapsto g_r)\big)(w) &= \tau_g(r \mapsto g_r)\left(\left.\frac{\partial r}{\partial s}\right|_{s=0}(w)\right) \\ &= \mathcal{T}\left(\left.\frac{\partial g_r}{\partial r}\right|_{r=0}\left(\left.\frac{\partial r}{\partial s}\right|_{s=0}(w)\right)\right).\end{aligned}$$

Thus, $\tau_g(s \mapsto r \mapsto g_r) = [s \mapsto r]^*\big(\tau_g(r \mapsto g_r)\big)$. Therefore, τ_g is a differential 1-form pointed at g.

Next, let $g'_r = f_r^*(g_r)$. We have

$$\begin{aligned}\left.\frac{\partial g' r}{\partial r}\right|_{r=0}(v) &= \left.\frac{\partial f_r^*(g_r)}{\partial r}\right|_{r=0}(v) \\ &= \left.\frac{\partial f_r^*(g)}{\partial r}\right|_{r=0}(v) + \left.\frac{\partial g_r}{\partial r}\right|_{r=0}(v).\end{aligned}$$

Now let $\phi_s(x) = f_{sv}(x)$, s real. Then,

$$\left.\frac{\partial f_r^*(g)}{\partial r}\right|_{r=0}(v) = \left.\frac{\partial \phi_s^*(g)}{\partial s}\right|_{s=0},$$

and defining the vector field ξ by

$$\xi(x) = \left.\frac{\partial \phi_s^*(x)}{\partial s}\right|_{s=0},$$

we get:

$$\left.\frac{\partial f_r^*(g)}{\partial r}\right|_{r=0}(v) = \delta_{\mathrm{L}} g \quad \text{with} \quad \delta \mathrm{x} = \xi(x).$$

Thus,

$$\left.\frac{\partial g' r}{\partial r}\right|_{r=0}(v) = \left.\frac{\partial g_r}{\partial r}\right|_{r=0}(v) + \delta_{\mathrm{L}} g.$$

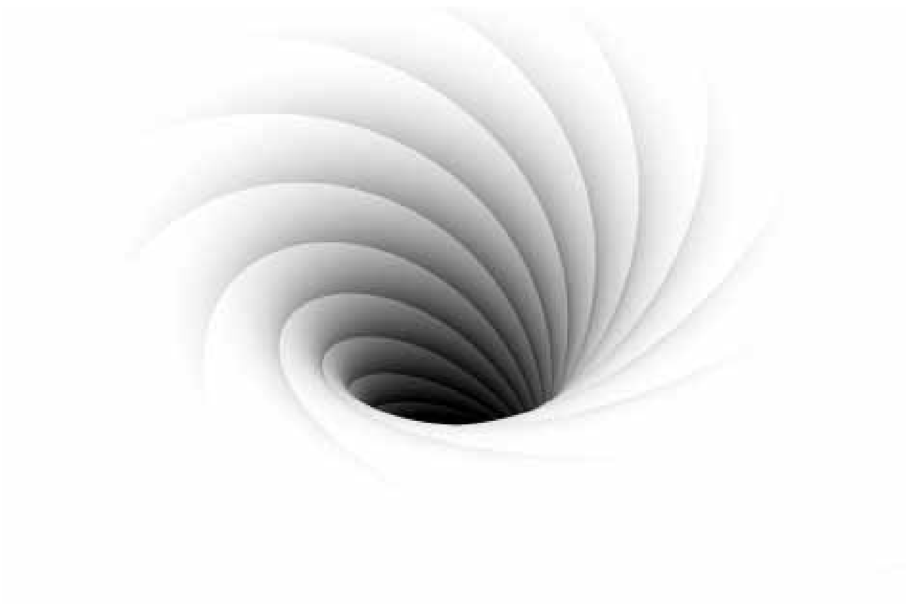
Hence:

$$\mathcal{T}(\delta g') = \mathcal{T}(\delta g) + \mathcal{T}(\delta_{\mathrm{L}} g),$$

where

$$\delta g = \left.\frac{\partial f_r^*(g)}{\partial r}\right|_{r=0}(v).$$

Therefore, the 1-form τ_g, pointed at g, descends to the quotient if $\mathcal{T}$ is Eulerian. ▶



Postface: The Beginning of Diffeological Spaces

The word "diffeologies" first appears in Souriau's paper *Groupes différentiels*, the text of a talk presented at a conference in Salamanca in 1979 and published in 1980 [**Sou80**]. In this article, a "difféologie" refers to an abstract structure used to define exclusively what Souriau called "groupes différentiels". A "difféologie" is defined by five axioms, the last two of which relate specifically to the group structure. Although Souriau could have removed the last two axioms from this list, and thus defined the general concept of "espaces différentiels", we note that he did not do so in this article and only considered his "difféologies" in the context of groups.

The general concept of "espace différentiel", as it would later emerge, is in fact absent from this 1980 paper. We shall see that it would take three years, and some unexpected developments, for this notion to appear in its own right.

Actually, Souriau's focusing on groups is somewhat understandable when we know that he was looking at this time for a renewal of his method of quantization through the Gelfand-Naimark-Segal construction,[1] that is, using positive-definite functions on a given group of symmetries to build a Hilbert space and a unitary representation. What he suggested then was a mean to conditionally connect these representations, depending on some positive function, with an appropriate coadjoint orbit, and thus achieve the Dirac quantization program. But because the Dirac program requires to represent the

[1]Not to mention Souriau's profound bias for groups vis-à-vis general spaces.

whole group of symplectomorphisms,[2] Souriau had to leave the category he used to work in, that is, the category of finite dimensional Lie groups, to deal with the huge infinite dimensional group of all the symplectomorphisms of a symplectic manifold. A group about which Kostant told him once: "It is too big".

Souriau knew very well that dealing with infinite dimensional groups is a serious challenge, especially if you don't want to just be heuristic but mathematically correct. He knew also that he didn't need to involve a too sophisticated structure on the group of symplectomorphisms to get what he was looking for. Just a few "differentiable" properties would be enough. Then, instead of paying tribute to the thick theory of topological groups, he preferred to build his own light category of groups that he called "groupes différentiels", by retaining just the minimal properties he needed for his purpose. And that gave his paper on "Groupes différentiels", with its five axioms, which we are talking about [**Sou80**].

Then, during a couple of years Souriau continued to work on this approach and tried to figure out a way to fulfill Dirac program of quantization in this context.

At the time, Paul Donato, also a student of Souriau, was interested in this new concept of "groupe différentiel". He wrote an essay on the characterization of the fundamental group of a manifold through its group of diffeomorphisms, published in 1981 [**Don81**], and continued to investigate the fundamental group of groups of diffeomorphisms of manifolds and more generally the fundamental group, and coverings, of cosets of "groupes différentiels". Eventually, this made the subject of his dissertation that he defended later in 1984 [**Don84**]. On my side, I was working on something very different, the classification of SO(3) symplectic manifolds, and these "groupes différentiels" did not speak to me that much.

[2]Originally infinitesimally.

The next step in the theory of "groupes différentiels" happened in June 1983 during a conference in Lyon,[3] where Souriau was invited to talk and Paul and I were attending. Souriau talked again about his new theory of "groupes différentiels" applied to the quantization program, he exposed his new developments. At the same conference we heard also a series of talks concerning what we called later "irrational tori", that is, the quotients of the 2-torus by an irrational line. This question was related to the study of quasi-periodic potentials in quantum physics. Alain Connes had already introduced his "non-commutative geometry" to study these objects that escape the traditional differential geometry.

At that moment Paul had already developed some tools to compute the fundamental group of a coset of a "groupe différentiel" and even to built its universal covering in some cases. We decided then to investigate the "irrational torus" from the "groupe différentiel" point of view. We computed its fundamental group and universal covering, for each irrational number. We were happy to find non trivial results, but we were really amazed when we found that two irrational numbers gave two diffeomorphic tori if and only if they were conjugate modulo $\mathrm{GL}(2, \mathbf{Z})$, we couldn't have hoped for better, but we could have get worse. We were even more surprised by the computation of the connected components of the group of diffeomorphisms of irrational tori that distinguish between the quadratic and non-quadratic irrational numbers. We published these results, in a preprint titled "Exemple de groupes différentiels : flots irrationnels sur le tore", in July 1983 [**DI83**]. As far as I know it is in this preprint that, for the first time, the expression "espace différentiel" was used. It appears between quotes, at the second page of the preprint (page 112 of the CPT preprints page numbering), in the sentence:

> Les applications différentiables de $\mathcal{T}_\alpha$ dans un "espace différentiel" E sont les applications $\varphi : \mathcal{T}_\alpha \to E$ telles que...

[3] *Feuilletages et Quantification Géométrique* Journées Lyonnaises de la SMF, 14-17 juin 1983.

On the other hand, the formal (axiomatic) definition of "espace différentiel" appeared a couple of months later, precisely in October 1983, in the Souriau preprint of the Lyon conference talk [**Sou84**]. I do remember that during this period we had some lively discussions about the need for such a general notion. In particular, on a summer's day that year, we were together, the whole team, at the cafeteria of Luminy campus for a coffee break. We were talking about diffeologies. We, Paul and I, insisted on a formal definition of differential spaces, and Souriau put up strong resistance because, at the time, he thought that such a generality was useless or at least irrelevant.[4] We could not agree because, for his ScD thesis, Paul was in need of a formal framework of "espaces différentiels" at least to introduce coherently his "espaces différentiels homogènes" [**Don84**]. On my side, following our paper with Paul, I began to think on a generalization of our results through general homotopy and fiber bundles, and for that I needed a general theory of "espaces différentiels" as fundation. Indeed, homotopy and fiber bundles in diffeology became my ScD thesis I defended in 1985, where "espaces différentiels" became "espaces difféologiques" [**Igl85**].[5] Under this pressure, Souriau finally changed his mind and included this definition a few weeks later, in his October preprint.

This is how the diffeology of spaces was born, and we can finally ask why it took so long to move from groups to spaces and from five axioms to three.

[4]Indeed, if we have only in mind the needs of Geometric Quantization, involving just homogeneous spaces.

[5]Fixing that way a language incoherence highlighted by A.E. Van Est.

Appendix: A Categorical Approach to Diffeology

Enxin Wu
Department of Mathematics
Shantou University, Guangdong
P.R. China

From the very beginning, diffeology was introduced from a geometric point of view. But one can take an abstract categorical approach to understand and study this subject. In this short appendix, we briefly discuss this alternative aspect, and see how to organize the basic theory of diffeology, especially the categorical viewpoint of some geometric constructions and homotopy theory in diffeology.

A bit category theory is needed to understand this appendix, as the standard terminologies are not defined here. The first section requires the basics of categories, functors, natural transformations, limits, colimits and cartesian closedness; The next two sections use adjunctions and (left) Kan extensions. Some familiarity with the corresponding contents in [**McL71**] suffices.

Diffeological spaces versus concrete sheaves

The classical sheaf theory can be viewed as making the mathematical formalism of the local-to-global principle with underlying topological spaces. For an example, to define a suitable continuous map from a complicated topological space to **R**, one could first decompose the space into simpler pieces (eg. an open cover), and define suitable continuous maps from these pieces to **R**. Then one could try to patch them together to get a global one if it is possible. This idea can be formalized rigorously using category theory as follows:

DEFINITION 1. *For a topological space* X*, write* $\mathcal{O}_X$ *for all open subsets of* X*, ordered by inclusions. A (set-valued)* ***presheaf*** *on* X *is a functor* $\mathcal{O}_X^{op} \to \mathfrak{Set}$ *from the opposite category of* $\mathcal{O}_X$ *to the category* $\mathfrak{Set}$ *of sets and functions. A (set-valued)* ***sheaf*** *on* X *is a (set-valued) presheaf* $\mathcal{F}$ *on* X *such that for any* $U \in \mathcal{O}_X$ *and any open covering* $U = \bigcup_i U_i$*, we have an equalizer diagram in* $\mathfrak{Set}$*:*

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{j,k} \mathcal{F}(U_j \cap U_k)$$

where the two parallel arrows are induced by the inclusions $U_j \cap U_k \to U_j$ *and* $U_j \cap U_k \to U_k$*, respectively.*

It is easy to observe that if one abstracts the essential properties of $\mathcal{O}_X$ out of the above definition, one could define sheaves over other nice categories:

DEFINITION 2. *A* ***presheaf*** *on an arbitrary category* $\mathcal{A}$ *is a functor* $\mathcal{A}^{op} \to \mathfrak{Set}$.

A ***Grothendieck topology*** *Cov on a category* $\mathcal{A}$ *is an assignment of a collection* $Cov(A)$ *of collections of morphisms with the common codomain* A *for each object* A *of* $\mathcal{A}$ *such that*

* $\{1_A : A \to A\} \in Cov(A)$*;*
* *If* $A' \to A$ *is a morphism in* $\mathcal{A}$ *and* $\{A_i \to A\}_i \in Cov(A)$*, then each pullback* $A' \times_A A_i$ *exists in* $\mathcal{A}$*, and* $\{A' \times_A A_i \to A'\}_i \in Cov(A')$*;*
* *If* $\{A_i \to A\}_i \in Cov(A)$*, and* $\{A_{ij} \to A_i\}_j \in Cov(A_i)$ *for each i, then* $\{A_{ij} \to A_i \to A\}_{i,j} \in Cov(A)$.

An element in $Cov(A)$ *is called a* ***covering*** *of* A.

A ***site*** *is a category with a Grothendieck topology.*

A ***sheaf*** *on a site* $(\mathcal{A}, Cov)$ *is a presheaf* $\mathcal{F} : \mathcal{A}^{op} \to \mathfrak{Set}$ *on* $\mathcal{A}$ *such that for each covering* $\{A_i \to A\}_{i \in I}$ *of* A*, we have an equalizer diagram in* $\mathfrak{Set}$*:*

$$\mathcal{F}(\mathrm{A}) \longrightarrow \prod_{i\in \mathrm{I}} \mathcal{F}(\mathrm{A}_i) \rightrightarrows \prod_{j,k\in \mathrm{I}} \mathcal{F}(\mathrm{A}_j \times_{\mathrm{A}} \mathrm{A}_k)$$

where the two parallel arrows are induced by the canonical maps $\mathrm{A}_j \times_{\mathrm{A}} \mathrm{A}_k \to \mathrm{A}_j$ *and* $\mathrm{A}_j \times_{\mathrm{A}} \mathrm{A}_k \to \mathrm{A}_k$*, respectively.*

EXAMPLE 3. Let X be a topological space, and let $\mathcal{O}_{\mathrm{X}}$ be the poset of all open subsets of X together with inclusions. For any open subset U of X, write Cov(U) for the collection of the usual open coverings of U. This gives rise to a Grothendieck topology on $\mathcal{O}_{\mathrm{X}}$, and the above two definitions of sheaves on $\mathcal{O}_{\mathrm{X}}$ coincide.

EXAMPLE 4. First example: Let $\mathcal{E}^\infty$ be the category of all open subsets of $\mathbf{R}^n$ for all $n \in \mathbf{N} = \{0, 1, 2, 3, \ldots\}$ and smooth (i.e., infinitely differentiable) maps between them. For any object U of $\mathcal{E}^\infty$, write Cov(U) for the collection of the usual open coverings of U. This gives rise to a Grothendieck topology on $\mathcal{E}^\infty$.

Second example: In a similar manner, if we only change the morphisms of $\mathcal{E}^\infty$ to be C^k (i.e., k-times continuously differentiable) for $k \in \mathbf{N}$ or C^ω (i.e., analytic), then we again get sites denoted $\mathcal{E}^k$ and $\mathcal{E}^\omega$, respectively.

Third example: Similarly, all open subsets of $\mathbf{C}^n$ for all $n \in \mathbf{N}$, all holomorphic maps between them, and the usual open coverings together form a site, denoted $\mathcal{C}^\omega$.

From a differential geometer's point of view, the general sheaves are abstract to work with. It would be nice if there is an underlying set which dominates all the structures. This leads to the following definition:

DEFINITION 5. *A* ***concrete site*** $(\mathcal{A}, Cov)$ *is a site with a terminal object* 1 *such that*

* *the functor* $\mathrm{Hom}_{\mathcal{A}}(1, \cdot) : \mathcal{A} \to \mathfrak{Set}$ *is faithful;*
* *for every object* A *in* $\mathcal{A}$ *and every covering* $\{\mathrm{A}_i \to \mathrm{A}\}_i \in Cov(\mathrm{A})$*, the induced map* $\coprod_i \mathrm{Hom}_{\mathcal{A}}(1, \mathrm{A}_i) \to \mathrm{Hom}_{\mathcal{A}}(1, \mathrm{A})$ *is surjective.*

A ***concrete sheaf*** $\mathcal{F}$ *over a concrete site* $(\mathcal{A}, Cov)$ *with a terminal object* 1 *is a sheaf over* $(\mathcal{A}, Cov)$ *such that the natural map*

$$\mathcal{F}(\mathrm{A}) \to \mathrm{Hom}_{\mathfrak{Set}}(\mathrm{Hom}_{\mathcal{A}}(1, \mathrm{A}), \mathcal{F}(1))$$

defined by $a \in \mathcal{F}(A) \mapsto (f \in \mathrm{Hom}_{\mathcal{A}}(1, A) \mapsto f^*(a) \in \mathcal{F}(1))$*, is injective for every object* A *of* $\mathcal{A}$*.*

We simply write $\mathfrak{CSh}(\mathcal{A})$ *for the category of concrete sheaves over the concrete site* $(\mathcal{A}, Cov)$ *and natural transformations between them.*

EXAMPLE 6. All of the sites $\mathcal{E}^\infty, \mathcal{E}^k, \mathcal{E}^\omega, \mathcal{C}^\omega$ are concrete. They share a common terminal object which consists of a single point. The rest axioms for the concreteness are clear since every object (resp. morphism, covering) in each of these sites has a set-theoretical meaning.

EXAMPLE 7. The site $\mathcal{O}_X$ is concrete if and only if every covering of any object U in $\mathcal{O}_X$ is essentially the trivial covering $1_U : U \to U$.

Here are some analogous definitions to that of a diffeological space and a smooth map between diffeological spaces:

DEFINITION 8. *Given a concrete site* $(\mathcal{A}, Cov)$ *with a terminal object* 1*, write* $\underline{A}$ *for the set* $\mathrm{Hom}_{\mathcal{A}}(1, A)$ *for each object* A *of* $\mathcal{A}$*. An* $\mathcal{A}$***-space** *is a set* X *together with a collection* $\mathcal{S}$ *of maps (called the* ***structure maps****)* $\underline{A} \to X$ *of sets for all objects* A *of* $\mathcal{A}$ *such that the following three axioms hold:*

* *Every constant map* $\underline{A} \to X$ *is in* $\mathcal{S}$*;*
* *If* $\underline{A} \to X \in \mathcal{S}$ *and* $A' \to A$ *is a morphism in* $\mathcal{A}$*, then the composite* $\underline{A'} \to \underline{A} \to X$ *is in* $\mathcal{S}$*;*
* *Given a map* $f : \underline{A} \to X$*, if there is a covering* $\{A_i \to A\}_i$ *of* A *in* $Cov(A)$ *such that each composite* $\underline{A_i} \to \underline{A} \to X$ *is in* $\mathcal{S}$*, then so is the map* f*.*

A map $X \to Y$ *between two* $\mathcal{A}$*-spaces is called an* $\mathcal{A}$***-map****, if it is a function of the two underlying sets, which sends every structure map of* X *to a structure map of* Y*.* $\mathcal{A}$*-spaces and* $\mathcal{A}$*-maps form a category, denoted* $\mathcal{A}$*-Spaces.*

THEOREM 9. *For a concrete site* $(\mathcal{A}, Cov)$*, the two categories* $\mathfrak{CSh}(\mathcal{A})$ *and* $\mathcal{A}$*-Spaces are equivalent.*

Proof. Let 1 be a terminal object of the site. Define $F : \mathfrak{CSh}(\mathcal{A}) \to \mathcal{A}$-Spaces by sending a concrete sheaf $\mathcal{F}$ over $\mathcal{A}$ to $\mathcal{F}(1)$; the structure maps on $\mathcal{F}(1)$ consist of all $\mathcal{F}(A)$ (viewed as a subset of $\mathrm{Hom}_{\mathfrak{Set}}(\underline{A}, \mathcal{F}(1))$ from the definition of a concrete sheaf) for all objects A in $\mathcal{A}$; a morphism $\mathcal{F} \to \mathcal{G}$ is sent to the induced map $\mathcal{F}(1) \to \mathcal{G}(1)$.

Define $G : \mathcal{A}\text{-Spaces} \to \mathfrak{CSh}(\mathcal{A})$ by sending an $\mathcal{A}$-space X to a concrete sheaf $\mathcal{F}_X$ such that $\mathcal{F}_X(A)$ = the collection of structure maps of X of the form $\underline{A} \to X$, for every object A of $\mathcal{A}$; an $\mathcal{A}$-map $f : X \to Y$ in $\mathcal{A}$-Spaces is sent to a natural transformation $\mathcal{F}_X \to \mathcal{F}_Y$ induced by post-composition with f.

By unravelling the definitions of a concrete sheaf and a natural transformations between two concrete sheaves, and the definitions of an $\mathcal{A}$-space and an $\mathcal{A}$-map between two $\mathcal{A}$-spaces, one sees that F and G are functors which give rise to the equivalence of the two categories.[1] □

THEOREM 10. *Let $(\mathcal{A}, Cov)$ be a concrete site. Then*

* *$\mathcal{A}$ is canonically a fully faithful subcategory of $\mathfrak{CSh}(\mathcal{A})$.*
* *The category $\mathfrak{CSh}(\mathcal{A})$ is bicomplete and cartesian closed.*
* *Every concrete sheaf over a concrete site $\mathcal{A}$ is a colimit of a diagram of $\mathcal{A}$ in $\mathfrak{CSh}(\mathcal{A})$.*

Proof. We can replace $\mathfrak{CSh}(\mathcal{A})$ by the category $\mathcal{A}$-Spaces.

(1) Define $e : \mathcal{A} \to \mathcal{A}\text{-Spaces}$ by sending an object A of $\mathcal{A}$ to an $\mathcal{A}$-space $e(A)$ with the underlying set $\underline{A}$. And $\underline{A'} \to \underline{A}$ is defined to be a structure map if and only if it is induced by a morphism $A' \to A$ in $\mathcal{A}$. A morphism $A \to A''$ in $\mathcal{A}$ then induces an $\mathcal{A}$-map $\underline{A} \to \underline{A''}$. Therefore, e is a functor. Moreover, two distinct morphisms $A \to A''$ in $\mathcal{A}$ induces two distinct $\mathcal{A}$-maps $e(A) \to e(A'')$ since $1_A : A \to A$ can distinguish them. Furthermore, every $\mathcal{A}$-map $e(A) \to e(A'')$ is induced from a morphism $A \to A''$ in $\mathcal{A}$, again by using the structure map of $e(A)$ induced by $1_A : A \to A$. These prove the first result.

(2) Let $F : \mathcal{I} \to \mathcal{A}\text{-Spaces}$ be a functor from a small category $\mathcal{I}$. Write $U : \mathcal{A}\text{-Spaces} \to \mathfrak{Set}$ for the forgetful functor. Then $\lim F$ is an $\mathcal{A}$-space which has $\lim(U \circ F)$ as the underlying set, and $\underline{A} \to \lim F$ is a structure map if and only if each composite $\underline{A} \to \lim F \to F(i)$ is a structure map

[1]We will need to use the fact that $\underline{A' \times_A A''} \cong \underline{A'} \times_{\underline{A}} \underline{A''}$ in $\mathfrak{Set}$ in the proof, and this fact can be easily verified by the universal property of pullback.

of $F(i)$ for each object i of $\mathcal{I}$. Similarly, $\operatorname{colim} F$ is an $\mathcal{A}$-space which has $\operatorname{colim}(U \circ F)$ as the underlying set. And $\underline{A} \to \operatorname{colim} F$ is a structure map if and only if there is a covering $\{A_j \to A\}_j$ of A in $\mathrm{Cov}(A)$ such that for each j there is an object i in $\mathcal{I}$ and a structure map $\underline{A_j} \to F(i)$ making the following diagram commutative:

$$\begin{array}{ccccc} & & & & F(i) \\ & & & \nearrow & \downarrow \\ \underline{A_j} & \longrightarrow & \underline{A} & \longrightarrow & \operatorname{colim} F \end{array}$$

Given two $\mathcal{A}$-spaces X and Y, simply write $\mathcal{A}(X, Y)$ for the collection of all morphisms $X \to Y$. Define $f : \underline{A} \to \mathcal{A}(X, Y)$ to be a structure map if and only if the adjoint $\tilde{f} : e(A) \times X \to Y$ is an $\mathcal{A}$-map, where the domain is the product of two $\mathcal{A}$-spaces and hence again an $\mathcal{A}$-space as we have just proved that the category $\mathcal{A}$-Spaces is complete. It is straightforward to check that $\mathcal{A}(X, Y)$ with these structure maps is an $\mathcal{A}$-space, and moreover, there is an adjoint pair of functors

$$\cdot \times Y : \mathcal{A}\text{-Spaces} \rightleftharpoons \mathcal{A}\text{-Spaces} : \mathcal{A}(Y, \cdot),$$

for any fixed $\mathcal{A}$-space Y.

(3) Given an $\mathcal{A}$-space X, write $\mathcal{A}/X$ for the category whose objects are structure maps $\underline{A} \to X$ and whose morphisms are commutative triangles

$$\begin{array}{ccc} \underline{A'} & \xrightarrow{\tilde{f}} & \underline{A} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array}$$

where p, q are structure maps of X and $\tilde{f}$ is induced from a morphism $f : A' \to A$ in $\mathcal{A}$. Let $\mathcal{A}/X \to \mathcal{A}$-Spaces be the functor sending the above commutative triangle to the $\mathcal{A}$-map $e(A') \to e(A)$ induced by f. In other words, this functor is a composition $\mathcal{A}/X \to \mathcal{A} \to \mathcal{A}$-Spaces. It is straightforward to check that the colimit of this functor is isomorphic to X as $\mathcal{A}$-spaces. □

COROLLARY 11. *First of all, $\mathfrak{CSh}(\mathcal{E}^\infty)$ is equivalent to the category $\mathfrak{Diff}$ of diffeological spaces.*

Then, $\mathfrak{Diff}$ is bicomplete and cartesian closed.

Next, every smooth manifold[2] *is a gluing of Euclidean domains via diffeomorphisms, while every diffeological space is a gluing of Euclidean domains via smooth maps.*

A bit more about concrete sheaves is discussed in [**BH09**].

Constructions in diffeology revisited

The general slogan is, some structures of $\mathcal{E}^\infty$ can be compatibly (hence functorially) pushed to diffeological spaces. This slogan will be realized via Kan extension of a functor $\mathcal{E}^\infty \to \mathcal{C}$ (resp. $(\mathcal{E}^\infty)^{\mathrm{op}} \to \mathcal{C}$) to some category $\mathcal{C}$ along the embedding $\mathcal{E}^\infty \to \mathfrak{Diff}$ (resp. $(\mathcal{E}^\infty)^{\mathrm{op}} \to (\mathfrak{Diff})^{\mathrm{op}}$) as follows.

D-topology. There is an adjoint pair of functors

$$D : \mathfrak{Diff} \rightleftharpoons \mathfrak{Top} : C$$

where the functor D sends a diffeological space X to a topological space D(X) with the same underlying set. The topology (called the **D-topology**) on D(X) consists of all subsets A of X such that $p^{-1}(A)$ is open in U with the Euclidean topology for every plot $p : U \to X$ of X;[3] the functor C sends a topological space Y to a diffeological space C(Y) with the same underlying set. The diffeology on C(Y) consists of all continuous maps $U \to Y$ for every Euclidean domain U with the Euclidean topology.

Note that every smooth map between diffeological spaces is continuous when both domain and codomain are equipped with the D-topology. Moreover, the left adjoint D preserves colimits. In particular, the D-topology of a quotient diffeological space is the quotient topology.

[2]By a smooth manifold, we always require it to be finite-dimensional, Hausdorff, second-countable and without boundary.

[3]More categorically, the functor D is the left Kan extension of the forgetful functor $\mathcal{E}^\infty \to \mathfrak{Top}$ along the embedding $\mathcal{E}^\infty \to \mathfrak{Diff}$.

As a warning, the D-topology does not behave well with limits. For example, the D-topology of a diffeological subspace contains the sub-topology, and the D-topology of a product of diffeological spaces contains the product topology of the D-topologies, but they can be different.

The D-topology was first introduced in [**Igl85**]. See [**CSW14, WZ22**] for more about the D-topology.

Differential forms and de Rham cohomology. For any $k \in \mathbf{N}$, there is a functor $\Omega^k : (\mathcal{E}^\infty)^{op} \to \mathfrak{Vect}$ sending a Euclidean domain U to the vector space[4] $C^\infty(U, \wedge^k \mathbf{R}^{\dim U})$ of the differential k-forms on U. The left Kan extension of $\Omega^k : (\mathcal{E}^\infty)^{op} \to \mathfrak{Vect}$ along the embedding $(\mathcal{E}^\infty)^{op} \to (\mathfrak{Diff})^{op}$ is the differential k-form functor (again denoted Ω^k) for diffeological spaces.

Moreover, since exterior derivative commutes with pullback of forms on $\mathcal{E}^\infty$, we have a functor $\Omega^* : (\mathcal{E}^\infty)^{op} \to \mathfrak{Ch}(\mathfrak{Vect})$ which sends a Euclidean domain to its cochain complex of differential forms. The left Kan extension of Ω^* along the embedding $(\mathcal{E}^\infty)^{op} \to (\mathfrak{Diff})^{op}$ sends every diffeological space to its cochain complex of differential forms, and its cohomology is called the de Rham cohomology.

Differential forms and de Rham cohomology for diffeological spaces were further systematically developed in [**PIZ10, TB**].

Tangent and cotangent bundles. It is a bit controversial about how to define tangent and cotangent bundles for diffeological spaces. There are several different approaches in the existing literatures. We choose the following one that fits closely to the slogan at the beginning of this section.

For a general diffeological space, we could imagine that its tangent (resp. cotangent) bundle may not be locally (eg. in the sense of D-topology) trivial. We might still wish to have tangent (resp. cotangent) bundles that are fibrewise vector spaces. This leads to the following definition:

DEFINITION 12. *A **diffeological vector pseudo-bundle** is a morphism* $\pi : E \to B$ *in* $\mathfrak{Diff}$ *such that*

[4]Indeed it is a diffeological vector space with the functional diffeology.

* *Each fibre $\pi^{-1}(b)$ is a vector space;*
* *The fibrewise addition $E \times_B E \to E$ and the fibrewise scalar multiplication $\mathbf{R} \times E \to E$ are both smooth;*
* *The zero section $B \to E$ is smooth.*

A ***bundle map*** *from one diffeological vector pseudo-bundle $\pi_1 : E_1 \to B_1$ to another $\pi_2 : E_2 \to B_2$ is a commutative diagram in $\mathfrak{Diff}$*

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ B_1 & \xrightarrow[g]{} & B_2 \end{array}$$

such that f restricted to each fibre $\pi_1^{-1}(b_1) \to \pi_2^{-1}(g(b_1))$ is linear.

Diffeological vector pseudo-bundles and bundle maps form a category, denoted $\mathfrak{Dvpb}$.

There is a functor $T : \mathcal{E}^\infty \to \mathfrak{Dvpb}$ sending a Euclidean domain U to its tangent bundle $TU \to U$. The left Kan extension of T along the embedding $\mathcal{E}^\infty \to \mathfrak{Diff}$ is defined to be the tangent bundle for diffeological spaces.

In a similar manner, for each $k \in \mathbf{N}$, there is a functor $\wedge^k T^* : (\mathcal{E}^\infty)^{op} \to \mathfrak{Dvpb}$ sending a Euclidean domain U to its k^{th} exterior bundle $\wedge^k T^*(U) \to U$. The left Kan extension of $\wedge^k T^*$ along the embedding $(\mathcal{E}^\infty)^{op} \to (\mathfrak{Diff})^{op}$ is defined to be the k^{th} exterior bundle for diffeological spaces, whose sections correspond to differential k-forms discussed above.

Other tangent or cotangent bundles for diffeological spaces can be found in [**Hec94, TB, Vin08, CW23**]. See [**CW16, CW23**] for some comparison and various calculated examples.

Homotopy theory for diffeological spaces

Iglesias' geometric homotopy theory. The first important and nontrivial example in diffeology, i.e., the irrational tori T_α, was calculated in [**DI83**]. This led to diffeological bundle theory and geometric homotpy theory developed in [**Igl85**]. The textbook already contains these material, so we will not repeat them here. We only quote one result as follows. From the long exact sequence of smooth homotopy groups of the diffeological bundle $\mathbf{R} \to T_\alpha$, we know that the smooth fundamental group of T_α is $\mathbf{Z} \oplus \mathbf{Z}$; while the continuous fundamental group of the topological space $D(T_\alpha)$ is 0.

An attempt of model structure by Christensen-Wu. About the two fundamental groups of the irrational tori example discussed above, how to homotopically "correct" this difference? This is an essential part of my Ph.D. work [**Wu12**], which is further refined in [**CW14**]. It provides a potential answer to this question, together with some geometric applications.

By choosing a cosimplicial object A^* in $\mathfrak{Diff}$, we get an adjoint pair

$$|?|_{A^*} : s\mathfrak{Set} \rightleftharpoons \mathfrak{Diff} : S_{A^*} \tag{A1}$$

where the left adjoint is given by the left Kan extension of $A^* : \Delta^{op} \to \mathfrak{Diff}$ along the embedding $\Delta^{op} \to s\mathfrak{Set}$.

To be consistent with the definition of diffeology and to be connected with the related construction in topology, we choose A^n to be the unbounded n-simplex in $\mathbf{R}^{n+1}$ as a subspace, i.e., $A^n = \{(x_0, x_1, \ldots, x_n) \in \mathbf{R}^{n+1} \mid \sum_{i=0}^n x_i = 1\}$,[5] together with the usual coboundary and codegeneracy maps.

Combining this adjunction with the adjunction

$$D : \mathfrak{Diff} \rightleftharpoons \mathfrak{Top} : C$$

discussed above, there is a very enlightening comparison with the classical geometric realization and associated simplicial set adjunction

$$|?| : s\mathfrak{Set} \rightleftharpoons \mathfrak{Top} : S \tag{A2}$$

which states that

[5]It is isomoprhic to $\mathbf{R}^n$ in $\mathfrak{Diff}$.

* For any simplicial set X, there is a weak equivalence between $D(|X|_{A^*})$ and $|X|$ in $\mathfrak{Top}$;
* For any topological space Y, there is a weak equivalence between $S_{A^*}(C(Y))$ and $S(Y)$ in $\mathfrak{sSet}$.

This motivates us to conceive that there is a Quillen model structure[6] on the category $\mathfrak{Diff}$ of diffeological spaces such that it is Quillen equivalent to the standard model structure on $\mathfrak{sSet}$ via the adjunction (A1) above (and hence to the standard model structure on $\mathfrak{Top}$ as well). We further conjecture that this model structure can be transferred from the standard model structure on $\mathfrak{sSet}$ to $\mathfrak{Diff}$ via the adjunction (A1).

As an easy observation, smoothness provides some rigidity via the inverse function theorem when comparing with continuity. In particular,

$$|\Lambda^n|_{A^*} \to |\Delta^n|_{A^*} \cong \mathbf{R}^n \tag{A3}$$

has no smooth retract[7]. If our conjecture holds, this would imply that not every diffeological space is fibrant or cofibrant. Furthermore, the potential answer to the question we proposed at the beginning of this part is, irrational tori T_α is fibrant but not cofibrant[8], so the cofibrant replacement $S^1 \times S^1 = T^2 \to T_\alpha$[9] gives a "correction" of the smooth fundamental group of T_α from the continous one.

[6]A large amount of homotopy theory in $\mathfrak{Top}$ relies on some interrelationship between Serre fibrations, Serre cofibrations and weak equivalences. These properties were extracted as axioms by Quillen in [**Qui67**], called a Quillen model structure. This approach unifies stable homotopy theory in topology, homological algebra in algebra, A^1-homotopy theory in algebraic geometry, etc. In particular, it provides an equivalence of the combinatorial model of homotopy theory from $\mathfrak{sSet}$ to the geometric model of homotopy theory from $\mathfrak{Top}$ via the adjunction (A2) above, called a Quillen equivalence. The importance here is, there shall also be a smooth model of homotopy theory from $\mathfrak{Diff}$.

[7]Indeed, it is not appropriate to call it a "retract" here, since $|\Lambda^n|_{A^*}$ does not have the subset diffeology of $\mathbf{R}^n$. The fact is, even if we equip it with the subset diffeology, there is still no smooth retract, which is implied by the inverse function theorem.

[8]This part has been proved in [**CW14**].

[9]This relates to another conjecture that every smooth manifold is cofibrant in this model structure on $\mathfrak{Diff}$.

Note that the domain $|\Lambda^n|_{A^*}$[10] of (A3) is the union of all coordinate hyperplanes in $\mathbf{R}^n$ with the gluing diffeology. Here is a geometric question: For which diffeological spaces X, does every smooth map $|\Lambda^n|_{A^*} \to X$ extend to a smooth map $\mathbf{R}^n \to X$ via the map (A3), for all $n \in \mathbf{N}$? This is equivalent to the question that when is the simplicial set $S_{A^*}(X)$ a Kan complex?[11] If X is a diffeological abelian group, then it is not hard to write down an explicit formula for the extension. But in general, this is not straightforward. We used some simplicial set theory to get several classes of such diffeological spaces. In particular, we gave three different proofs for smooth manifolds.

Unfortunately, the conjecture was recently disproved in [**Pav22**].

A model structure by Kihara. At the mean time, in preprints posted on Arxiv, Haraguchi and Shimakawa proposed a model structure on Diff which is Quillen equivalent to the standard model structure on $\mathfrak{Top}$. The first attempts had some flaws, but we can hope that they will be addressed in the future. Nevertheless, a key idea, there, is a modification of the subset diffeology of $\mathbf{R}$ on the closed interval (and higher dimensional analogues) to avoid the rigidity from the inverse function theorem. Then mimic the standard cofibrantly generated model structure on $\mathfrak{Top}$ to prove the model axioms.

In [**Kih19**], Kihara got a model structure on $\mathfrak{Diff}$ which is Quillen equivalent to the standard model structure on $\mathfrak{sSet}$. The idea is a kind of mixing of Christensen-Wu and Haraguchi-Shimakawa. In slightly more detail, he uses an alternative cosimplicial object $\mathbf{B}^*$ to connect $\mathfrak{sSet}$ and $\mathfrak{Diff}$. It is the standard bounded one as the underlying sets, with the diffeology recursively defined, so that $|\Lambda^n|_{\mathbf{B}^*} \to |\Delta^n|_{\mathbf{B}^*}$ is an induction which also has a smooth retract for every $n \geq 2$. Then mimic the standard cofibrantly generated model structure on $\mathfrak{sSet}$ to prove the model axioms. Equivalently, Kihara successfully lifts the standard model structure on $\mathfrak{sSet}$ to $\mathfrak{Diff}$ via the adjunction

$$|?|_{\mathbf{B}^*} : \mathfrak{sSet} \rightleftharpoons \mathfrak{Diff} : S_{\mathbf{B}^*}.$$

[10] $|\Lambda^n_k|_{A^*}$ are isomorphic in $\mathfrak{Diff}$ for all $k = 0, 1, \ldots, n$, simply denoted by $|\Lambda^n|_{A^*}$.
[11] Such X is (called) fibrant in [**CW14**] when related to our main conjecture.

This model structure was further applied in [**Kih23**] to study homotopy theory of infinite-dimensional manifolds.

Final remarks

There are more material related to the categorical (and homotopical) approach towards diffeology. We will have to omit them due to limited space and time. These include Bunk's, Pavlov's and Schreiber's approaches to homotopy theory of (simplicial) (pre)sheaves over some subsites of $\mathcal{E}^\infty$; Iglesias', Krepski-Watts-Wolbert's, and Ahmadi's approaches to diffeological sheaves and Cěch cohomologies; Watts' and Miyamoto's work on diffeological groupoids; Iwase's work on diffeological CW complexes and loop spaces; Kuribayashi's work on rational homotopy theory, etc. The interested reader shall be able to find most of them on the web.

As a final warning, the reader should check the existing results before applying them, no matter they are published or not, as there could be some mistakes.

email: exwu@stu.edu.cn

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