When Diffeology Meets Noncommutative Geometry

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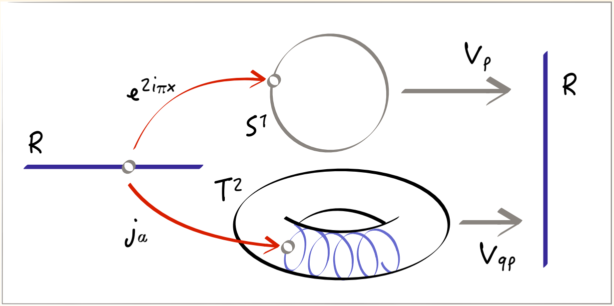
Geometry Seminar
Shantou University March 31, 2025

Outline.

- A bit of history.
- The objects: orbifolds, quasifolds.
- The diffeology through atlas and strict generating family.
- The Structural Groupoid associated with an atlas:
  - Local smooth maps and their functional diffeology
  - Germification
  - Extraction of the structural groupoid
- The theorem on transitive component (little lemma)
- Different atlases give equivalent groupoids.
- Definition of the $\mathbf{C}^*$-algebra associated with an atlas
- Mulhy-Renault-Williams Equivalence
- Different atlasses give MRW-equivalent groupoids
- MRW-equivalent groupoids give Morita equivalent algebras
- Diffeomorphic quasifolds have Morita equivalent algebras

In the 1980s emerges the quantization of quasi-periodic potential. Problem of the spectrum of the Hamiltonian operator of a particle on a line submitted to a quasi periodic potential.

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$



A bit of history I

- Alain Connes, **C***-*algebres et géométrie différentielle.* In C.R. Acad. Sci. Paris Sér. A-B 290 (1980).
  ☞ The algebra A_θ: Algebra of irrational rotation.
- Jean-Marie Souriau, *Groupes différentiels.* Lect. Notes in Math. 836 (1980)
  ☞ Groups of diffeomorphisms: infinite dimensional groups.
- Paul Donato and Patrick Iglesias, *Exemple de groupes différentiels : flots irrationnels sur le tore.* Preprint CPT-83/P.1524, Centre de Physique Théorique, Marseille (1983)
  ☞ The irrational torus T_α: Singular space.

The Objective.

Establish the relationship between
diffeology and non-commutative geometry.

(The case of Orbifolds/Quasifolds)

{Diffeology} $\supset$ {Quasifolds} → Groupoids → $\mathbf{C}^*$-Algebras

Exemple de groupes différentiels : flots irrationnels sur le tore, by Paul Donato & Patrick Iglesias Preprint CPT-83/P.1524. 1983. & C. R. Acad. Sci. 301(4), Paris (1985).

Noncommutative Geometry & Diffeology: The Case Of Orbifolds, by Patrick Iglesias-Zemmour & Jean-Pierre Laffineur. Journal of Noncommutative Geometry. vol. 12, No 4, 2018, pp. 1551–1572.

Quasifolds, Diffeology and Noncommutative Geometry, by Patrick Iglesias-Zemmour & Elisa Prato. Journal of Noncommutative Geometry, vol. 15, No 2, 2021, pp. 735–759.

A (new) survey ➠ *Why Diffeology*, Preprint (2025). `http://math.huji.ac.il/~piz/documents/WD.pdf`.

Diffeology

What is a Diffeology

A diffeology on a set X declares which parametrizations should be regarded as differentiable, or smooth.

This subset $\mathcal{D}$ of parametrizations should satisfy three axioms:

- Covering axiom.
- Locality axiom.
- Smooth compatibility axiom.

A parametrization in a set X is any map from an Euclidean domain $U \subset \mathbf{R}^n$, $n \in \mathbf{R}$, into X.

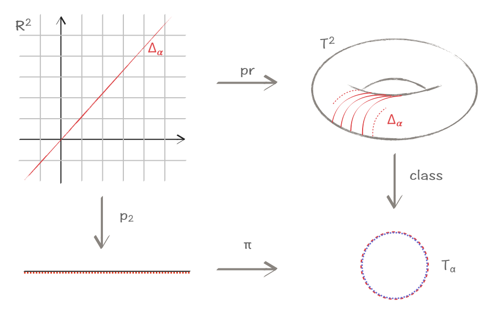
With diffeologies come smooth maps between diffeological spaces that make the caregory {Diffeology} a complete, cocomplete, Cartesian closed category.

First Example — The Irrational Torus I

The Irrational torus,[1] which appeared in 1983, was the first significant diffeological space that drew attention to diffeology.

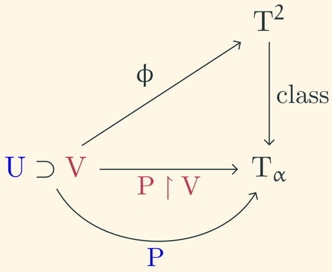
$$T_\alpha = T^2/\Delta_\alpha$$

with $\alpha \in \mathbf{R} - \mathbf{Q}$.



[1] Paul Donato & P.I-Z. *Exemple de groupes différentiels : flots irrationnels sur le tore.* Preprint CPT-83/P.1524. (1983). In C. R. Acad. Sci, 301(4), Paris (1985).

The Plots in T_α



Facts on T_α

With diffeology, we can continue to do differential geometry on spaces with trivial topology like the irrational torus T_α. Here are some facts:

Fact 1. The irrational torus T_α is diffeomorphic to the quotient $\mathbf{R}/(\mathbf{Z}+\alpha\mathbf{Z})$.

Fact 2. The projection $\pi : \mathbf{R} \to \mathrm{T}_\alpha \simeq \mathbf{R}/(\mathbf{Z}+\alpha\mathbf{Z})$ is its universal covering, unique up to an isomorphism.

Fact 3. The first homotopy group of T_α identifies with $\mathbf{Z}+\alpha\mathbf{Z} \subset \mathbf{R}$.

Facts on T_α

Fact 4. T_α and T_β are diffeomorphic if they are conjugate by an unimodular transformation. That is, if there exists

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Z}) \text{ such that } \beta = \frac{a\alpha + b}{c\alpha + d}.$$

Fact 5. The diffeomorphisms of T_α are the projections of maps $x \mapsto \lambda x + \mu$, where μ is any real number and λ belongs to a subgroup of the multiplicative non-zero real number which identifies with the group of components of $\mathrm{Diff}(T_\alpha)$, when $\mathrm{Diff}(T_\alpha)^0 = T_\alpha$.

$$\pi_0(\mathrm{Diff}(T_\alpha)) \simeq \begin{cases} \{\pm 1\} \times \mathbf{Z}, & \text{if } \alpha \text{ is a quadratic number;} \\ \{\pm 1\} & \text{otherwise.} \end{cases}$$

Orbifolds & Quasifolds

Diffeological Orbifold

An Orbifold is a diffeological space locally diffeomorphic at each point to some $\mathbf{R}^n/\Gamma$, with Γ finite.

Definition. [IKZ10][2] A n-orbifold is a diffeological space X such that: for every point $x \in X$ there exist :

- A finite subgroup $\Gamma \subset GL(n, \mathbf{R})$.
- A local diffeomorphism f from $\mathbf{R}^n/\Gamma$ to X such that $x \in f(U)$, with $U = \mathrm{dom}(f)$. The diffeomorphism f is a Chart of X. A set of charts covering X is an Atlas.

Originally introduced as V-Manifolds by I. Satake [IS56, IS57].

[2] P.I-Z, Yael Karshon & Moshe Zadka. *Orbifolds as diffeology,* Transactions of the AMS, 362, no. 6, (2010), p 2811-2831

The Teardrop I

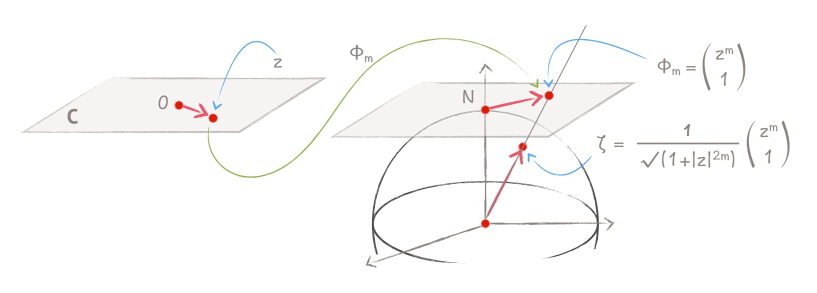
The plots giving S^2 a teardrop structure of Orbifold:

$$\zeta : U \to S^2 \subset \mathbf{C} \times \mathbf{R}, \text{ with } \zeta(r) = \begin{pmatrix} z(r) \\ t(r) \end{pmatrix}, |z(r)|^2 + t(r)^2 = 1,$$

- if $\zeta(r_0) \neq N$, then there exists a small ball $\mathcal{B}$ centered at r_0 such that $\zeta \restriction \mathcal{B}$ is smooth.
- If $\zeta(r_0) = N$, then there exist a small ball $\mathcal{B}$ centered at r_0 and a smooth parametrization z in $\mathbf{C}$ defined on $\mathcal{B}$ such that, for all $r \in \mathcal{B}$,

$$\zeta(r) = \frac{1}{\sqrt{1 + |z(r)|^{2m}}} \begin{pmatrix} z(r)^m \\ 1 \end{pmatrix}.$$

The diffeology of the Teardrop summarized:



Diffeological Orbifold: Advantage of Diffeology.

In the original definition, Satake was unable to give a satisfactory notion of smooth maps between orbifolds. Indeed, in [IS57, page 469], he writes this footnote:

> *"The notion of* C^∞*-map thus defined is inconvenient in the point that a composite of two* C^∞*-maps defined in a different choice of defining families is not always a* C^∞ *map."*

By embedding orbifolds in the category {Diffeology}, they become a subcategory with morphisms: the smooth maps in the sense of diffeology. And that resolved Satake's problem.

Diffeological Quasifolds

Quasifolds have been originally introduced by Elisa Prato in 2001[3] as a generalization of orbifolds.

They appear naturally in a new domain: Nonrational toric geometry.

We get a diffeological version as follow:

DEFINITION. A quasifold is a diffeological space locally diffeomorphic at each point to some $\mathbf{R}^n/\Gamma$, where $\Gamma \subset \mathrm{Aff}(\mathbf{R}^n)$ is countable.

[3] E. Prato. *Simple Non–Rational Convex Polytopes via Symplectic Geometry.* Topology, **40**, pp. 961–975 (2001).

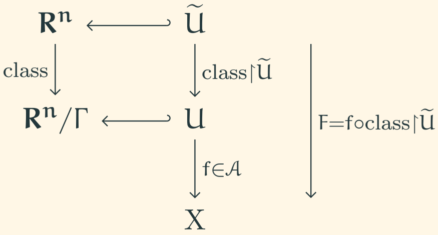
Structure Groupoid

Strict Generating Family

The structure groupoid of a quasifold is built from strict generating families.

DEFINITION. Let X be an n-quasifold, $\mathcal{A}$ an atlas, f a chart and $\mathrm{class} : \mathbf{R}^n \to \mathbf{R}^n/\Gamma$ be the projection, then :

- $F = f \circ \mathrm{class}$ is a plot of X called the strict lifting of f
- $\mathcal{F} = \{F \mid f \in \mathcal{A}\}$ is a strict generating family of X



Strict Generating Family Picture

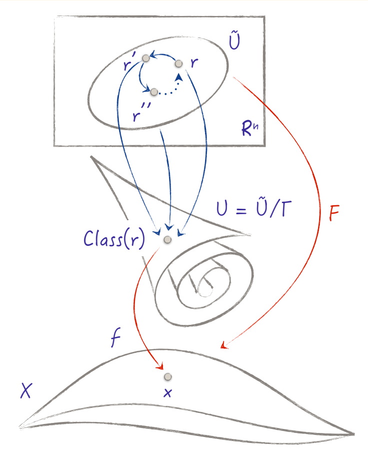


Figure 1: The three levels of a generating family.

The Structure Groupoid of a Quasifold

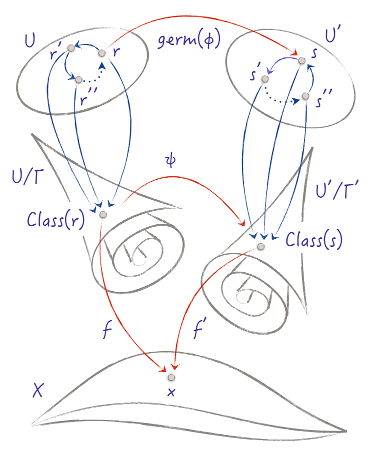
➠ X an n-quasifold, $\mathcal{A}$ an atlas, $\mathcal{F}$ the strict generating family over $\mathcal{A}$ and $\mathcal{N}$ the nebula:

$$\mathcal{N} = \coprod_{F \in \mathcal{F}} \mathrm{dom}(F).$$

$$\mathrm{ev} : \mathcal{N} \to X \quad \text{with} \quad \mathrm{ev}(F, r) = F(r).$$

DEFINITION. The Structure Groupoid $\mathbf{G}$ of the quasifold X is the subgroupoid of germs of local diffeomorphisms of $\mathcal{N}$ which project onto the identity of X along ev. That is,

$$\mathrm{Mor}_{\mathbf{G}}((F, r), (F', r')) = \left\{ \mathrm{germ}(\varphi)_r \;\middle|\; \begin{array}{l} \varphi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbf{R}^n), r' = \varphi(r) \\ F' \circ \varphi = F \restriction \mathrm{dom}(\varphi) \end{array} \right\}$$



The three levels of a quasifold.

The Diffeology on the Structure Groupoid

Local Smooth Maps.

The diffeology of the structure groupoid starts with the functional diffeology on local smooth maps.

DEFINITION. Let X and X′ be two diffeological spaces, $\mathrm{f} : \mathrm{X} \supset \mathrm{A} \longrightarrow \mathrm{X}'$ is local smooth if for all $\mathrm{P} \in \mathcal{D}$,

$$\mathrm{P}' = \mathrm{f} \circ \mathrm{P} \in \mathcal{D}'.$$

- Note that that implies $\mathrm{P}^{-1}(\mathrm{A})$ is open.
- $\mathcal{C}^\infty_{\mathrm{loc}}(\mathrm{X}, \mathrm{X}')$: the set of local smooth maps from X to X′.

$$\begin{array}{ccc} \mathrm{X} \supset \mathrm{A} & \xrightarrow{\ \mathrm{f}\ } & \mathrm{X}' \\ \uparrow{\scriptstyle \mathrm{P}} & & \uparrow{\scriptstyle \mathrm{P}'} \\ \mathcal{U} & \hookleftarrow & \mathrm{P}^{-1}(\mathrm{A}) \end{array}$$

Diffeology on the Local Smooth Maps I

➠ X and X' are two diffeological spaces.

- $\mathfrak{F} = \{(f, x) \mid f \in \mathcal{C}^\infty_{\mathrm{loc}}(X, X') \text{ and } x \in \mathrm{dom}(f)\}$.
- Ev : $\mathcal{C}^\infty_{\mathrm{loc}}(X, X') \times X \supset \mathfrak{F} \to X'$ with $\mathrm{Ev}(f, r) = f(r)$.

PROPOSITION. There exists a coarsest diffeology on $\mathcal{C}^\infty_{\mathrm{loc}}(X, X')$ such that Ev is a local smooth map. This diffeology is called the functional diffeology.

PROPOSITION. Composition of local smooth maps is smooth for the functional diffeology.

➠ $P : r \mapsto f_r$ be a parametrization in $\mathcal{C}^\infty_{\mathrm{loc}}(X, X')$ defined on U, and

$$\mathcal{U} = \{(r, x) \mid x \in \mathrm{dom}(f_r)\} \subset U \times X\}$$

PROPOSITION. P is a plot of the functional diffeology if $\mathrm{Ev}_P : (r, x) \mapsto f_r(x)$, defined on $\mathcal{U} \subset U \times X$ is local smooth.

DEFINITION. For $P : U \to \mathrm{Diff}_{\mathrm{loc}}(X)$, P is a plot of the (pseudogroup) functional diffeology if $r \mapsto f_r$ and $r \mapsto (f_r)^{-1}$ are two plots of the functional diffeology.

DEFINITION. The Groupoid $\mathbf{G}$ of Germs of Local Diffeomorphisms is defined by:

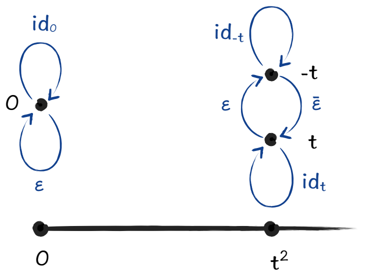
$$\begin{cases} \mathrm{Obj}(\mathbf{G}) & = & X, \\ \mathrm{Mor}(\mathbf{G}) & = & \{\, \mathrm{germ}(\varphi)_x \mid \varphi \in \mathrm{Diff}_{\mathrm{loc}}(X) \text{ and } x \in \mathrm{dom}(\varphi)\}. \end{cases}$$

- $\mathrm{src}(\mathrm{germ}(\varphi)_x) = x$, $\mathrm{trg}(\mathrm{germ}(\varphi)_x) = \varphi(x)$.
- $\mathrm{germ}(\varphi)_x \cdot \mathrm{germ}(\varphi')_{x'} = \mathrm{germ}(\varphi' \circ \varphi)_x$, with $x' = \varphi(x)$.
- Let $\begin{cases} \mathfrak{G} = \{(\varphi, x) \mid \varphi \in \mathrm{Diff}_{\mathrm{loc}}(X) \text{ and } x \in \mathrm{dom}(\varphi)\}. \\ \mathrm{germ} : (\varphi, x) \mapsto \mathrm{germ}(\varphi)_x. \end{cases}$

➠ We equip $\mathrm{Mor}(\mathbf{G})$ with the pushforward of the diffeology of $\mathfrak{G}$ by the map germ.

THEOREM. The fibers of the subduction $\mathrm{ev} : \mathrm{Obj}(\mathbf{G}) \to X$, $\mathrm{ev}(F, r) = F(r)$, are exactly the transitivity components of $\mathbf{G}$.

For example:



The groupoid, of the orbifold $\mathbf{R}/\{\pm 1\}$.

According to the definition of equivalence of groupoids[4]

Corollary. Different atlases of X give equivalent groupoids.

Proof. Let $\mathcal{A}$ and $\mathcal{A}'$ be two atlases. Let $\mathcal{A}'' = \mathcal{A} \coprod \mathcal{A}'$. Let $\mathbf{G}, \mathbf{G}'$ and $\mathbf{G}''$ be the associated groupoids. Then, $\mathbf{G}$ and $\mathbf{G}'$ are two full and faithful subgroupoids which have the same space of groupoid components, that is X. □

➠ The equivalence class of the structure groupoids of an orbifold is a diffeological invariant.

[4] Mac Lane. *Category For The Working Mathematician*, Chap. 4 § 4 Thm. 1

General description of the Structure Groupoid

➠ $\mathbf{G}$ the structure groupoid of a quasifold X. Set theoretically:

$$\mathbf{G} = \coprod_{x \in X} \mathbf{G}_x \text{ with } \begin{cases} \mathrm{Obj}(\mathbf{G}_x) = \mathrm{ev}^{-1}(x) \\ \mathrm{Mor}(\mathbf{G}_x) = (\mathrm{ev} \circ \mathrm{src})^{-1}(x). \end{cases}$$

➠ $\mathbf{G}_x$ is the assemblage of the elementary subgroupoids $\mathbf{G}_x^F$.

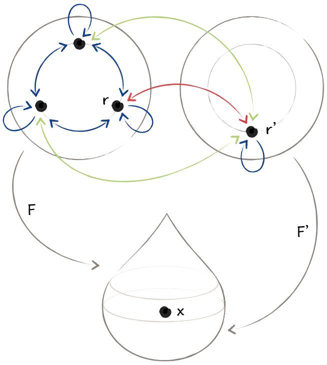
$$\begin{cases} \mathrm{Obj}(\mathbf{G}_x^F) = \{F\} \times \mathrm{dom}(F), \\ \mathrm{Mor}(\mathbf{G}_x^F) = \{\mathrm{germ}(\varphi)_r \in \mathrm{Mor}(\mathbf{G}_x) \mid r, \varphi(r) \in \mathrm{dom}(F)\}. \end{cases}$$

Then, with the F_i the charts such that $x \in \mathrm{val}(F_i)$:

$$\mathbf{G}_x = \mathbf{G}_x^{F_1} — \mathbf{G}_x^{F_2} — \cdots — \mathbf{G}_x^{F_{N_x}}.$$

General description of the Structure Groupoid

Because it is easier to draw for an orbifold than a general quasifold, here is the example of the teardrop:



Assembling the Groupoid of the Teardrop

Elementary Structure Groupoid : The Little Lemma

THEOREM. The arrows of the groupoid $\mathbf{G}_x^F$ are the germs of the diffeomorphisms $r \mapsto \gamma \cdot r$, where $r \in \mathrm{dom}(F)$ and $\gamma \in \Gamma$.

➠ That is, the groupoid $\mathbf{G}_x^F$ is the groupoid of the action of the local symmetries of the quasifold.

Note 1. The isotropy group of $r \in \mathrm{dom}(F)$ is the stabilizer of r in Γ. Since they are all isomorphic independantly of the atlas, their type defines the Isotropy of $x = F(r)$.

Note 2. For quasifolds of type $X = U/\Gamma$, with Atlas $\{\mathbf{1}_U\}$, it is not necessary to consider all this construction since in this case the groupoid is just the action of Γ.

Proof. Let $\pi : \mathbf{R}^n \to \mathbf{R}^n/\Gamma$, let φ in $\mathrm{Diff}_{\mathrm{loc}}(\mathbf{R}^n)$, defined on a ball $\mathcal{B}$ centered at r_0, s.t. $\pi \circ \varphi = \varphi$. Then, for all r there exists $\gamma \in \Gamma$ s.t. $\varphi(r) = \gamma \cdot r$. Next, the map $\varphi_\gamma : \mathcal{B} \to \mathbf{R}^n \times \mathbf{R}^n$, $\varphi_\gamma(r) = (\varphi(r), \gamma \cdot r)$ is smooth. The set of $\Delta_\gamma = \varphi_\gamma^{-1}(\Delta)$, where Δ is the diagonal in $\mathbf{R}^n \times \mathbf{R}^n$, is a closed covering of $\mathcal{B}$. By Baire's theorem there exists at least one of the Δ_γ with a non empty interior. By recursion one has:

Lemma. The union $\cup_{\gamma \in \Gamma} \mathring{\Delta}_\gamma$ is a dense open subset of $\mathcal{B}$.

By differentiability of φ and because $\Gamma \subset \mathrm{GL}(n, \mathbf{R})$ (or $\mathrm{Aff}(\mathbf{R}^n)$) is discrete, there exists only one $\gamma \in \Gamma$ such that $D(\varphi)(\mathcal{B}) = \{\gamma\}$. And then, $\varphi(r) = \gamma \cdot r$. ☐

Noncomutative Geometry

Let $\mathbf{G}$ be a topological groupoid, $\mathcal{C}_c(\mathbf{G})$ the compactly supported continuous complex fonctions on $\mathrm{Mor}(\mathbf{G})$.

DEFINITION. The convolution and the involution are defined by

$$f * g(\gamma) = \sum_{\beta \in \mathbf{G}^x} f(\beta \cdot \gamma) g(\beta^{-1}) \quad \text{and} \quad f^*(\gamma) = f(\gamma^{-1})^*,$$

where $x = \mathrm{src}(\gamma)$, and z^* is the conjugate of $z \in \mathbf{C}$. The sums involved are supposed to converge. The completion of the vector space $\mathcal{C}_c(\mathbf{G})$ for the uniform norm, equipped with these operations, is by definition[5] the $\mathbf{C}^*$-algebra associated with $\mathbf{G}$.

[5] Jean Renault. *A groupoid approach to* $\mathbf{C}^*$*–Algebras*. Lecture notes in Mathematics (793), Springer-Verlag Berlin Heidelberg New-York (1980).

$\mathbf{C}^*$-Algebra of a Quasifold

The main property of this construction is given by the following definition and theorem:

DEFINITION. Let X be a quasifold and $\mathcal{A}$ be an atlas. The structure groupoid $\mathbf{G}$ associated with $\mathcal{A}$ is Hausdorff and étale[6]. We get then a $\mathbf{C}^*$-Algebra associated with every atlas of X.

THEOREM. The $\mathbf{C}^*$-algebras associated with two different atlases are Morita equivalent. Therefore, diffeomorphic quasifolds define Morita equivalent $\mathbf{C}^*$-algebras.

[6] That is, the projection $\mathrm{src} : \mathrm{Mor}(\mathbf{G}) \to \mathrm{Obj}(\mathbf{G})$ is étale.

Sketch of Proof of the Theorem I

The theorem is a consequence of the Muhly-Renault-Williams theorem[7] that states that equivalent groupoids, in their sense, give Morita equivalent $\mathbf{C}^*$-algebras.

[7] Paul Muhly, Jean Renault and Dana Williams. Equivalence And Isomorphism For Groupoid $\mathbf{C}^*$-Algebras. J. Operator Theory 17, no 1 pp. 3–22 (1987).

Muhly-Renaud-Willian Equivalence

Let $\mathbf{G}$ and $\mathbf{G}'$ be two locally compact groupoids.

DEFINITION. We say that a locally compact space Z is a Muhly-Renaud-Willian $(\mathbf{G}, \mathbf{G}')$-equivalence if

(i) Z is a left principal $\mathbf{G}$-space.

(ii) Z is a right principal $\mathbf{G}'$-space.

(iii) The $\mathbf{G}$ and $\mathbf{G}'$ actions commute.

(iv) The action of $\mathbf{G}$ on Z induces a bijection of $Z/\mathbf{G}$ onto $\mathrm{Obj}(\mathbf{G}')$.

(v) The action of $\mathbf{G}'$ on Z induces a bijection of $Z/\mathbf{G}'$ onto $\mathrm{Obj}(\mathbf{G})$.

Muhly-Renaud-Willian Equivalence

Let X be a quasifold, $\mathbf{G}$ and $\mathbf{G}'$ be the groupoids associated to 2 atlases $\mathcal{A}$ and $\mathcal{A}'$, with strict generatingfamilies $\mathcal{F}$ and $\mathcal{F}'$.

We define Z to be the space of germs of local diffeomorphisms, from the nebula $\mathcal{N}$ of the family $\mathcal{F}$ to the nebula $\mathcal{N}'$ of the family $\mathcal{F}'$, that project on the identity by the evaluation map. That is,

$$Z = \left\{ \mathrm{germ}(f)_r \;\middle|\; \begin{array}{l} f \in \mathrm{Diff}_{\mathrm{loc}}(\mathrm{dom}(F), \mathrm{dom}(F'), r \in \mathrm{dom}(F), \\ F \in \mathcal{F}, F' \in \mathcal{F}' \text{ and } F' \circ f = F \restriction \mathrm{dom}(f). \end{array} \right\}.$$

The proof consists then to show that Z satisfies the conditions of a MRW-equivalence.

The simplest (non trivial) orbifold

EXAMPLE. Let $\Delta_1 = \mathbf{R}/\{\pm 1\}$. The $\mathbf{C}^*$-algebra $\mathcal{A}$ associated has the following representation into $M_2(\mathbf{C}) \otimes \mathcal{C}(\mathbf{R}, \mathbf{C})$:

$$M(t) = \begin{pmatrix} a(t) & b(-t) \\ b(t) & a(-t) \end{pmatrix} \text{ and } M^*(t) = [{}^{\tau}M(t)]^*,$$

with $a(t) = f(t, 1)$ and $b(t) = f(t, -1)$.

NOTE. The characteristic polynomial

$$P(\lambda) = [t \mapsto \lambda^2 - (a(t) + a(-t))\lambda + a(t)a(-t) - b(t)b(-t)],$$

of $M(t)$, is a smooth function defined on the orbifold Δ_1.

The Case of the Irrational Torus

The irrational torus T_α is also difeomorphic to the quotient

$$\mathrm{T}_\alpha \simeq \mathrm{S}^1/\mathbf{Z} \text{ with } m(z) = e^{2i\pi m\alpha}.$$

the groupoid $\boldsymbol{S}_\alpha$ of this action of $\mathbf{Z}$ on $\mathrm{S}^1 \subset \mathbf{C}$ is:

$$\mathrm{Obj}(\boldsymbol{S}_\alpha) = \mathrm{S}^1 \text{ and } \mathrm{Mor}(\boldsymbol{S}_\alpha) = \{(z, e^{2i\pi\alpha m}) \mid z \in \mathrm{S}^1 \text{ and } m \in \mathbf{Z}\}.$$

Its algebra $\mathfrak{A}_\alpha$ has been computed numerous times and is called[8] irrational rotation algebra. It is the universal $\mathbf{C}^*$-algebra generated by two unitary elements U and V, satisfying the relation $\mathrm{VU} = e^{2i\pi\alpha}\mathrm{UV}$.

[8] Marc A. Rieffel. $\mathbf{C}^*$–*Algebras Associated With Irrational Rotations.* Pacific Journal of Mathematics, Vol. 93, No. 2, 1981.

Proposition. If α and β are conjugate modulo $GL(2, \mathbf{Z})$, T_α and T_β are diffeomorphic[9]. Thus, $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\beta$ are Morita equivalent.

This is a diffeological proof of the direct sense of Rieffel's theorem 4 (*op. cit.*). So, diffeology can be used in noncommutative geometry as a link between geometry and algebra.

[9] Paul Donato & P.I-Z. *Exemple de groupes différentiels : flots irrationnels sur le tore.* Preprint CPT-83/P.1524. Centre de Physique Théorique, Marseille, 1983. In C. R. Acad. Sci, 301(4), Paris (1985).

Paul Donato & Patrick Iglesias. *Exemple de groupes différentiels : flots irrationnels sur le tore,* Preprint CPT-83/P.1524. Centre de Physique Théorique, Marseille, July 1983. In C. R. Acad. Sci. 301(4), Paris (1985).

Saunders Mac Lane *Categories for the working mathematician,* Graduate Text in Math, no. 5, Springer-Verlag (1978)

Patrick Iglesias-Zemmour, Yael Karshon, Moshe Zadka *Orbifolds as diffeology,* Transactions of the AMS, 362, no. 6, (2010), p 2811-2831

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- Jean Renault, *A Groupoid Approach to C*-Algebras*, Lectures Notes in Math., vol. 793, Springer Verlag, 1980
- Ishiro Satake. *On a Generalization of the Notion of Manifold*, Proceedings of the National Academy of Sciences **42** (1956), 359–363.
- —— *The Gauss-Bonnet Theorem for V-manifolds*. Journal of the Mathematical Society of Japan, **9**:4 (1957), 464–492.