Why Diffeology?

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Ouline

- Preamble — Why Diffeology
- The Axiomatic
- The Irrational Torus
- Fiber Bundles
- Homotopy
- Differential Calculus
- Singularities
- Noncommutative geometry
- Symplectic Diffeology
- General Relativity

Why Diffeology?

Why Diffeology? Because we need

- A simple, rigororous and powerful framework that goes beyond the natural limits of differential geometry, wether they are singularities, infinite dimension etc.
- Which can treat on the same footing: quotients, parts, products and sums.
- And even spaces of paths and, more generally, spaces of smooth maps.
- Which captures and preserves the unique characteristics of singular constructions.

Strategy of Diffeology

- Diffeology overcomes these limitations by abandoning the local model.
- The smoothness of a space is no longer associated with the existence of local charts,
- But only with the choice of a family of parametrizations into the space, which reveal its smooth structure.

Just as basal metabolism represents the minimal energy required for life, diffeology provides the minimal structure required for a meaningful theory of smoothness. In this sense it can be regarded as a basal framework for differential geometry.

The Axiomatic

Category Diffeology I

A diffeology on a set X is given by a choice $\mathcal{D}$ of Euclidean parametrizations (maps from Euclidean domains into X) called plots, satisfying 3 axioms.

- The plots cover X.
- To be a plot is local.
- The composite of a plot by a smooth parametrization of its domain is a plot.

A set X equipped with a diffeology $\mathcal{D}$ is a diffeological space.

- A map $f\colon X \to X'$, where X and X' are two diffeological spaces, is said to be smooth if the composite of f with a plot of X is a plot of X'.

Category Diffeology II

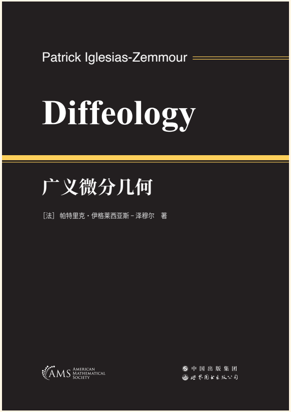
Diffeological spaces together with smooth maps define the category {Diffeology}. Isomorphisms are called diffeomorphisms. This category is stable for the set theoretic operations:

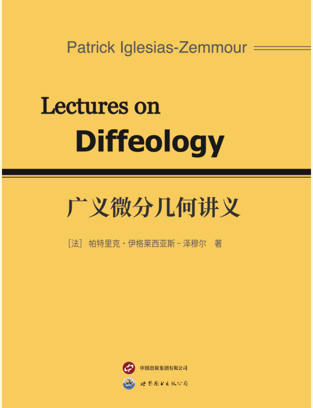
- Sums: $\coprod_i X_i$
- Subsets: $A \subset X$
- Quotients: $\mathcal{Q} = X/\sim$
- Products: $\prod_i X_i$

It is a complete and co-complete category.

The set $C^\infty(X, X')$ has a natural functional diffeology which makes {Diffeology} Cartesian closed.

Books on Diffeology





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The first significant example

- The irrational torus is the quotient of the 2-torus T^2 by the irrational flow

$$\mathcal{S}_\alpha = \left\{ \left(e^{2i\pi t}, e^{2i\pi\alpha t} \right) \mid t \in \mathbf{R} \right\} \subset \mathrm{T}^2 \quad \text{with} \quad \alpha \in \mathbf{R} - \mathbf{Q}.$$

- $\mathrm{T}_\alpha = \mathrm{T}^2/\mathcal{S}_\alpha$.
- Because α is irrational, $\mathcal{S}_\alpha$ is dense and T_α is topologically trivial.

The Irrational Torus II

The diffeology on T_α

The plots in T_α are the parametrizations

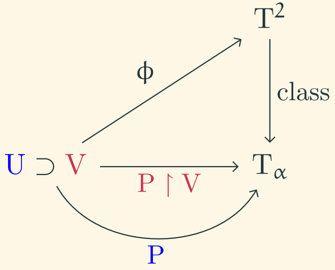
$$P : U \to T_\alpha,$$

that satisfy the condition:

- For all $r \in U$ there exists an open neighbourhood $V \subset U$ of r and a smooth map $\phi : V \to T^2$ such that: $P \upharpoonright V = \mathrm{class} \circ \phi$.

where $\mathrm{class} : T^2 \to T_\alpha$ is the projection.

The Plots in T_α



Facts on T_α

The diffeological space T_α is non-trivial, unlike its topology. Here are some facts:

Fact 1. The irrational torus T_α is diffeomorphic to the quotient $\mathbf{R}/(\mathbf{Z}+\alpha\mathbf{Z})$.

Fact 2. The projection $\pi : \mathbf{R} \to \mathrm{T}_\alpha \simeq \mathbf{R}/(\mathbf{Z}+\alpha\mathbf{Z})$ is its universal covering, unique up to an isomorphism.

Fact 3. The first homotopy group of T_α identifies with $\mathbf{Z}+\alpha\mathbf{Z} \subset \mathbf{R}$.

Facts on T_α

Fact 4. T_α and T_β are diffeomorphic if they are conjugate by an unimodular transformation. That is, if there exists

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Z}) \text{ such that } \beta = \frac{a\alpha + b}{c\alpha + d}.$$

Fact 5. The diffeomorphisms of T_α are the projection of affine maps $x \mapsto \lambda x + \mu$, where μ is a real number and λ belongs to a subgroup of the multiplicative non-zero real number which identifies with the group of connected components of the group of diffeommorphisms $\mathrm{Diff}(T_\alpha)$.

$$\pi_0(\mathrm{Diff}(T_\alpha)) \simeq \begin{cases} \{\pm 1\} \times \mathbf{Z}, & \text{if } \alpha \text{ is a quadratic number;} \\ \{\pm 1\} & \text{otherwise.} \end{cases}$$

The Irrational Torus VI

The homotopy of the irrational torus is the same as if the projection $\pi : T^2 \to T_\alpha$ was a fibration with fiber $\mathbf{R}$,

$$\begin{array}{ccc} & T^2 & \leftarrow \text{Total space} \\ \text{Fiber } \mathbf{R} \to & \Big\downarrow \pi & \\ & T_\alpha & \leftarrow \text{Base space} \end{array}$$

we could apply the exact long homotopy sequence, the homotopy of the base and because $\mathbf{R}$ is contractible. That would give: $\pi_0(T_\alpha) = \{T_\alpha\}$, $\pi_1(T_\alpha) = \mathbf{Z}^2$ and $\pi_k(T_\alpha) = 0$ for all $k > 1$,

But it is not since T_α is topologically trivial...

Fiber Bundles

Fiber Bundles in Diffeology

Fiber bundle will be defined in diffeology by relaxing the condition of local triviality:

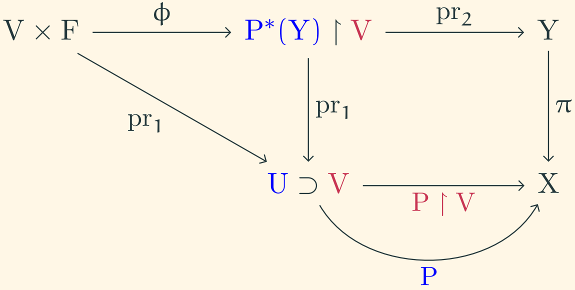
DEFINITION. A projection $\pi : Y \to X$ is a (diffeological) fibration, with fiber F, if, for every plot $P : U \to X$, the pullback

$$\mathrm{pr}_1 : P^*(Y) \to U \quad \text{with} \quad P^*(Y) = \{(r, y) \in U \times Y \mid P(r) = \pi(y)\},$$

is locally trivial with fiber F.

This is called local triviality along the plots.

The local structure of a diffeological fibration, for a plot $P : U \to X$ and $V \subset U$ is some open subset over which the pullback is trivial:



Facts on Fiber Bundles in Diffeology

Fact 1. A diffeological fiber bundle over a manifold is locally trivial.

Fact 2. Over a manifold, if the fiber is a manifold the total space of the fiber bundle is a manifold. Diffeological fibrations extend fully and faithfully theirs counterparts in ordinary differential geometry.

Fact 3. The projection of a diffeological group G over a quotient G/H, where H is any subgroup, is a diffeological fibration.

Fact 4. The projection class : $\mathrm{T}^2 \to \mathrm{T}_\alpha$ is a non trivial, non locally trivial,diffeological fibration.

Homotopy

Conectedness I

Homotopy if founded on the analysis of paths and loops. Let X be a diffeological space and $x \in X$.

$$\mathrm{Paths}(X) = C^\infty(\mathbf{R}, X),$$

$$\mathrm{Loops}(X, x) = \{\ell \in \mathrm{Paths}(X) \mid \hat{1}(\ell) = \hat{0}(\ell) = x\},$$

Where Paths(X) is equipped with the functional diffeology and Loops(X, x) with the subset diffeology.

Definition. Two points $x, x' \in X$ are connected, or homotopic, if they are the ends of some path γ: $\hat{0}(\gamma) = \gamma(0) = x$ and $\hat{1}(\gamma) = \gamma(1) = x'$, or $\mathrm{ends}(\gamma) = (x, x')$.

Connectedness II

Fact 1. Connectedness is an equivalence relation that divides the space into classes called connected components, or simply components. We denote

$$\pi_0(X) = X/\text{homotopy}$$

Fact 2. The decomposition into connected components is the finest that makes X the sum of its parts.

$$X = \coprod_{X_i \in \pi_0(X)} X_i.$$

Fundamental Group

Since the space of loops in X is a diffeological space, the construction of homotopy applies and the fundamental group of a diffeological space is defined by:

DEFINITION. The first group of homotopy of a (connected) space, also called fundamental group, if the space of components of $\mathrm{Loops}(X, x)$, based at the constant loop $\hat{x} = [t \mapsto x]$.

$$\pi_1(X, x) = \pi_0(\mathrm{Loops}(X, x), \hat{x}).$$

The multiplication is defined by the juxtaposition

$$\mathrm{comp}(\ell) \cdot \mathrm{comp}(\ell') = \mathrm{comp}(\ell \vee \ell').$$

Higher Homotopy Groups

The higher homotopy groups of X are defined by a recursion, for all $k \geq 1$:

$$\pi_k(X, x) = \pi_{k-1}(\text{Loops}(X, x), \hat{x}) \quad \text{and} \quad x_k = [t \mapsto x_{k-1}],$$

An equivalent presentation of the higher homotopy groups:

$$X_k = \text{Loops}(X_{k-1}, x_{k-1}) \quad \text{with} \quad x_k = [t \mapsto x_{k-1}],$$

And then

$$\pi_k(X, x) = \pi_1(X_{k-1}, x_{k-1}).$$

Example: $\pi_2(X, x) = \pi_1(X_1, x_1)$, $X_1 = \text{Loops}(X, x)$ and $x_1 = \hat{x}$, then $\pi_2(X, x) = \pi_0(\text{Loops}(X_1, x_1), \hat{x}_1) = \pi_0(\text{Loops}(\text{Loops}(X, x), \hat{x}), [t \mapsto [s \mapsto x]])$.

Homotopy of Fiber Bundles

Like their counterparts in ordinary differential geometry, diffeological fibrations $\pi : Y \to X$ satisfy the homotopy long exact sequence:

$$\cdots \to \pi_n(F, y) \to \pi_n(Y, y) \to \pi_n(X, x) \to \pi_{n-1}(F, y) \to \cdots$$
$$\cdots \to \pi_0(F, y) \to \pi_0(Y, y) \to \pi_0(X, x) \to 0,$$

where the fiber is chosen as $F = \pi^{-1}(x)$ and $y \in F$.

Applied to the irrational torus as the fibration $\text{class} : T^2 \to T_\alpha$ with contractible fiber $\mathbf{R}$, the sequence gives immediately the homotopy of T_α, i.e. $\pi_0(T_\alpha) = \{T_\alpha\}, \pi_1(T_\alpha) = \pi_1(T^2) = \mathbf{Z}^2$ and $\pi_k(T_\alpha) = \pi_k(T^2) = 0$ for all $k > 1$.

Differential Calculus

Differential Forms on Diffeological Spaces

- A differential k-form α on a diffeological space X associates with each plot P in X, a smooth form

$$\alpha(P) \in \Omega^k(\mathrm{dom}(P)),$$

such that, for all smooth parametrization F in dom(P)

$$\alpha(P \circ F) = F^*\big(\alpha(P)\big).$$

- Pullback: $f\colon X \to X'$ smooth, and $\alpha' \in \Omega^k(X')$, then,

$$\Omega^k(X) \ni f^*(\alpha')\colon P \mapsto f^*(\alpha')(P) = \alpha'(f \circ P).$$

Differential Calculus

Let X be a diffeological space and $\alpha \in \Omega^p(X)$. Then,

$$d\alpha \colon P \mapsto d[\alpha(P)]$$

is a differential $(p+1)$-form on X. The operator d is linear and smooth, for the functional diffeology. It satisfies

$$d \circ d = 0.$$

The vector space of closed k-forms, $d\alpha = 0$ is denoted by $Z^k(X)$.
The vector space of exact k-form, $\alpha = d\beta$ is denoted by $B^k(X)$.
The kth de Rham cohomology group (which is a vector space) is

$$H^k_{dR}(X) = Z^k(X)/B^k(X).$$

Chain Homotopy Operator

Connecting Forms on Space of Paths

There exists a smooth linear integro-differential operator

$$\mathcal{K} : \Omega^k(X) \to \Omega^{k-1}(\mathrm{Paths}(X)),$$

Satisfying

$$\mathcal{K} \circ d + d \circ \mathcal{K} = \hat{1}^* - \hat{0}^*.$$

That is, for all form α

$$\mathcal{K}[d\alpha] + d[\mathcal{K}\alpha] = \hat{1}^*(\alpha) - \hat{0}^*(\alpha).$$

Recall that $\hat{1}, \hat{0} : \mathrm{Paths}(X) \to X$, with $\hat{t}(\gamma) = \gamma(t)$.

Homotopic Invariance of De Rham Cohomology

THEOREM. *The pullback of a closed form by homotopic smooth maps are cohomologous.*

Proof. Let X and X′ be the diffeological spaces, and $t \mapsto f_t$ be a path in $C^\infty(X', X)$. let $F : X' \to \mathrm{Paths}(X)$ defined by:

$$F(x') = [t \mapsto f_t(x')].$$

Let $\alpha \in \Omega^k(X)$ and $d\alpha = 0$. Apply F^* to the Chain Homotopy Operator identity:

$$F^*(\mathcal{K}(d\alpha) + d[\mathcal{K}\alpha]) = F^*(\hat{1}^*(\alpha)) - F^*(\hat{0}^*(\alpha))$$

That gives $d[F^*(\mathcal{K}\alpha)] = f_1^*(\alpha) - f_0^*(\alpha)$, that is:

$$f_1^*(\alpha) = f_0^*(\alpha) + d\beta. \quad \square$$

Singularities

Singular Diffeological Spaces I

Examples of singular diffeological spaces:

- The irrational torus, because it D-topology is trivial and itself is not trivial.
- The half-line $[0, \infty[$ because 0 is distinguished by the topology.
- The quotient spaces $\mathbf{R}^n/\mathrm{O}(n)$ which look like $[0, \infty[$ but are they equivalent?
- Orbifolds and more generally quasifolds.
- More generally: every quotient, or subspace, of manifolds that is not a manifold.

Singular Diffeological Spaces II

Examples of singular diffeological spaces:

- What distinguishes the half-line $[0,\infty[\subset \mathbf{R}$ with the quotients its its dimension at 0:

$$\dim_0([0,\infty[) = \infty \qquad \dim_0 \mathbf{R}^n/\mathrm{O}(n) = n.$$

  So, by its specific definition of dimension, diffeology discriminate between the various quotients $\mathbf{R}^n/\mathrm{O}(n)$ and with the embedded half-line.
- Orbifolds have the same dimension everywhere (by construction) but have singular points: They cannot be exchanged by local diffeomorphisms.
- The irrational torus is homogeneous, it has the same dimension everywhere, but has a trivial D-topology.

The action of local diffeomorphisms

The singularities exhibited by the half-lines, as well by orbifolds or quasifolds are both captured by the action of local diffeomorphisms. That justifies the general introduction of the specific stratification proper to diffeology:

THEOREM. [Klein Stratification] *The partition of a diffeological space in orbits of local diffeomorphisms defines a stratification: it satisfies the frontier condition: The closure of an orbit, for the D-topology, is a union of orbits.*

Orbits are called Klein strata, they can be regarded as the definition of internal singularities. The dimension of the space is constant on Klein strata, that explains why in some cases the dimension captures the singularity.

Quotients of Manifolds

The Klein stratification is a natural and powerful diffeological invariant for the study of a quotient manifold M by a compact Lie group G. In the case of a symplectic manifold M^{2n} by an effective action of the torus T^n, the dimension map of the quotient space $Q_n = M^{2n}/T^n$ is related to the depth, added by hand to the topological structure of Q_n :[1]

$$\dim_x(Q_n) = \dim(M) - \mathrm{depth}(x).$$

For example $Q_n = \mathbf{C}^n/T^n$ has the D-topology of a corner $[0, \infty[^n$, but its dimension at x varies according to this formula.

[1] In particular in yet not published paper: *Classification of Locally Standard Torus Actions* by Yael Karshon and shintarô Kuroki.

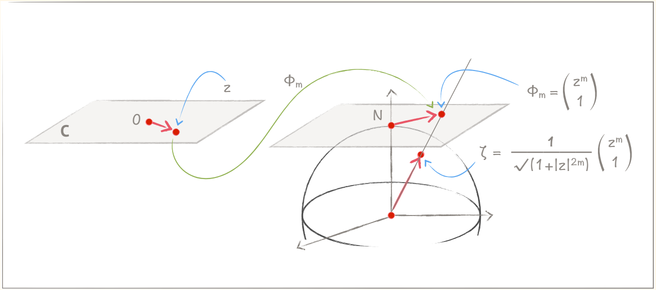
Orbifolds & Quasifolds

- An manifold is a diffeological space that is locally diffeomorphic to some $\mathbf{R}^n$.
- An orbifold is a diffeological space that is locally diffeomorphic to some $\mathbf{R}^n/\Gamma$, where Γ is a finite subgroup og the linear group $\mathrm{GL}(n, \mathbf{R})$, Γ may change with the point.
  Theorem. *This definition is equivalent to the original definition by Satake in his papers of 1956/57.*
- A quasifold is a diffeological space that is locally diffeomorphic to some $\mathbf{R}^n/\Gamma$, where Γ is a countable subgroup of the affine group $\mathrm{Aff}(n, \mathbf{R})$. This definition is an adaptation to diffeology of E. Prato original definition. The irrational torus is an example of quasifold.

The Plots of the TearDrop Orbifold

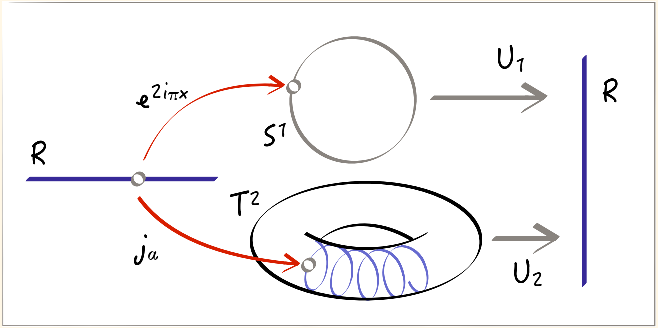
A plot ζ in $S^2 \subset \mathbf{C} \times \mathbf{R}$: If $\zeta(r_0) \neq N$, then there exists a ball $\mathcal{B}$ centered at r_0 such that $\zeta \restriction \mathcal{B}$ is smooth. If $\zeta(r_0) = N$, there exist a ball $\mathcal{B}$ centered at r_0 and a smooth parametrization z in $\mathbf{C}$ defined on $\mathcal{B}$ such that, for all $r \in \mathcal{B}$,

$$\zeta(r) = \frac{1}{\sqrt{1 + |z(r)|^{2m}}} \begin{pmatrix} z(r)^m \\ 1 \end{pmatrix}.$$



Diffeology and Noncommutative Geometry

The study of the irrational torus has been inspired by the involvment of quasi-periodic potentials in physics, and their mathematical treatment.



- This question has been adressed first by Alain Connes who introduced the concept of noncommutative geometry. The diffeological approach was a test (in the 1980s), with the irrational torus, to propose a pure geometric counterpart to this approach. But no structural links was made at this time between the two constructions.
- Much later in the 2010s we gave an categorical answer to this question: "How diffeology and noncommutative geometry are related?". This question has now a formal structural answer at least for orbifolds, and more generally for quasifolds.

$\mathbf{C}^*$-Algebras associated to Orbifolds & Quasifolds III

1. The connection between diffeology and noncommutative geometry is made according to Jean Renault's approach to noncommutative geometry by groupoids.

2. We associate to every generating family $\mathcal{F}$ of a quasifold Q (and then orbifold) a specific groupoid, and by Renault's construction a $\mathbf{C}^*$-algebra.

$\{$Diffeology$\} \supset \{$Quasifolds$\}$ → Groupoids → $\mathbf{C}^*$-Algebras

3. And prove then that different generating families define Morita equivalent $\mathbf{C}^*$-algebras, the natural equivalence in noncommutative geometry.

Structural groupoid of a Quasifolds

Let Q be a quasifold and $\mathcal{F}$ a generating family, their domains are open sets that factorize to Q through some quotient $\mathbf{R}^n/\Gamma$. Define then the groupoid F

- $\mathrm{Obj}(\mathsf{F}) = \coprod_{F \in \mathcal{F}} \mathrm{dom}(F)$, the nebula of $\mathcal{F}$.
- $\mathrm{Mor}_{\mathbf{G}}((F, r), (F', r'))$ is the set of local diffeomorphisms ϕ, from $\mathrm{dom}(F)$ to $\mathrm{dom}(F')$ such that $F(r) = F'(r')$ and $F' \circ \phi =_{\mathrm{loc}} F$, equipped with a functional diffeology of local smooth maps.

$$\begin{array}{ccc} \mathrm{dom}(F) \supset \mathrm{dom}(\phi) & \xrightarrow{\ \ \phi\ \ } & \mathrm{dom}(F') \\ & {\scriptstyle F} \searrow \quad \swarrow {\scriptstyle F'} & \\ & X & \end{array}$$

THEOREM 1. *The structural groupoids associated with two generating families are equivalent.*

The following convolution and the involution are defined on the space of compactly supported continuous complex functions on $\mathrm{Mor}(\mathsf{F})$:

$$f * g(\gamma) = \sum_{\beta \in \mathbf{G}^x} f(\beta \cdot \gamma) g(\beta^{-1}) \quad \text{and} \quad f^*(\gamma) = f(\gamma^{-1})^*,$$

THEOREM 2. *The space of compactly supported continuous complex functions on* $\mathrm{Mor}(\mathsf{F})$ *is a* $*$*-algebra that becomes* $\mathbf{C}^*$*-algebra after completion for the uniform norm.*

The main theorem of this construction is:

THEOREM 3. *Different generating families of a quasifold give Morita equivalent* $\mathbf{C}^*$*-algbras.*

COROLLARY. *Diffeomorphic quasifolds have Morita equivalent* $\mathbf{C}^*$*-algbras.*

The irrational torus is a typical example, for which the structural groupoid of the generating family $\{\text{class} : \mathbf{R} \to \mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z})\}$ is the groupoid of the $\mathbf{Z} + \alpha\mathbf{Z}$ action. This coincides with the noncommutative geometry construct.

Symplectic Diffeology

Diffeological Groups And Momenta

- A diffeological group is a diffeological space with a group law such that the product and the inverse are smooth.
- Every group of diffeomorphisms $\mathrm{Diff}(X)$ is a diffeological group, equipped with the functional diffeology.
- A momentum of a diffeological group G is a left-invariant 1-form. We introduce the vector space of momenta:

$$\mathcal{G}^* = \{\alpha \in \Omega^1(G) \mid \forall g \in G, L(g)^*(\alpha) = \alpha\}.$$

With $L(g)\colon g' \mapsto gg'$.

➠ It is important to notice the space of momenta $\mathcal{G}^*$ is not defined by duality with a presumed Lie algebra.

Symplectic Preamble

Symplectic geometry applied to mechanics has revealed two fundamental constructions:

- The group of automorphisms, the subgroup of Hamiltonian diffeomorphisms, Hamiltonian vector fields, one-parameter group of Hamiltonian diffeomorphisms, and so on.
- The moment map of groups of automorphisms, that capture a part (or the whole) of the geometry of the symplectic manifold.

These objects have a natural extension in diffeology, and we can build symplectic diffeology around these two constructions.[2]

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[2] I leave isotropic and co-isotropic subspaces for further investigations.

Moment Map: the **Simplest Case**

- Let us call parasymplectic form on a diffeological space X, any closed 2-form: $\omega \in \Omega^2(X)$ and $d\omega = 0$.
- Let G be a diffeological group acting smoothly on X and preserving ω: $\underline{g}^*(\omega) = \omega$, for all $g \in G$.

➠ Now assume that $\omega = d\alpha$ and $\underline{g}^*\alpha = \alpha$. Let $\hat{x} \colon g \mapsto \underline{g}(x)$ be the orbit map.

- The map $\mu \colon x \mapsto \hat{x}^*(\alpha)$, defined on X is smooth and takes its values in $\mathcal{G}^*$. This is the moment map of ω.

➠ Actually it is **<u>a</u>** moment map. The moment map associated with the primitive α.

Moment Map: the **General Case**

Let (X, ω) be a parasymplectic space, to get directly around the difficulty of the non-exactness of the closed 2-form ω, we use the chain-homotopy operator introduced previously

$$\mathcal{K} \colon \Omega^k(X) \to \Omega^{k-1}(\mathrm{Paths}(X)),$$

that satisfies the identity $d \circ \mathcal{K} + \mathcal{K} \circ d = \hat{1}^* - \hat{0}^*$. Define

$$\varpi = d\lambda, \text{ with } \lambda = \mathcal{K}\omega \text{ and } \varpi = \hat{1}^*(\omega) - \hat{0}^*(\omega).$$

If G preserves ω, then G preserve $\lambda = \mathcal{K}\omega$. And we are back to the simplest case $\underline{g}^*(\lambda) = \lambda$, but on the space of paths.

The Paths Moment Map

- The paths moment map is defined by:

$$\Psi\colon \mathrm{Paths}(X) \to \mathcal{G}^* \quad \text{with} \quad \Psi(\gamma) = \hat{\gamma}^*(\mathcal{K}\omega).$$

- It is G-equivariant for the coadjoint action. Let $g, k \in G$ and $\alpha \in \mathcal{G}^*$.

$$\mathrm{ad}(g)(k) = gkg^{-1}, \quad \mathrm{Ad}_*(g)(\alpha) = \mathrm{ad}(g)_*(\alpha).$$

Then

$$\Psi \circ g_* = \mathrm{Ad}_*(g) \circ \Psi \quad \text{with} \quad g_*(\gamma) = \underline{g} \circ \gamma.$$

- It is additive. For γ and γ' juxtaposable:

$$\Psi(\gamma \vee \gamma') = \Psi(\gamma) + \Psi(\gamma'),$$

The Two-Points Moment Map

- The two-points moment map, projection on $X \times X$ of the paths moment map, is defined by:

$$\psi\colon X \times X \to \mathcal{G}^*/\Gamma, \text{ with } \psi(x, x') = \Psi(\gamma),$$

with $x = \gamma(0)$, $x' = \gamma(1)$.

- $\Gamma \subset \mathcal{G}^*$ is made of Ad_*-invariant momenta. It it the Holonomy of the action of G on (X, ω), the obstruction of the action of G for being Hamiltonian.

$$\Gamma = \{\Psi(\ell) \mid \ell \in \mathrm{Loops}(X)\}.$$

- ψ is still G-equivariant and it is a Chasles cocycle:

$$\psi(x, x') + \psi(x', x'') = \psi(x, x'').$$

The One-Point Moment Map

- A one-point moment map is a solution μ of

$$\psi(x, x') = \mu(x') - \mu(x) \text{ with } \mu\colon X \to \mathcal{G}^*/\Gamma.$$

That is:

$$\mu(x) = \psi(x_0, x) + c, \text{ where } x_0 \in X \text{ and } c \in \mathcal{G}^*/\Gamma.$$

- The moment map μ is θ-affine Ad_*-equivariant:

$$\mu(\underline{g}(x)) = Ad_*(\mu(x)) + \theta(g), \text{ with}$$
$$\theta(g) = \psi(x_0, \underline{g}(x_0)) - \Delta(c)(g),$$

$\theta \in H^1(G, \mathcal{G}^*/\Gamma)$, and $\Delta(c)(g)$ is the coboundary $Ad_*(g)(c) - c$.

Action of $C^\infty(M, \mathbf{R})$ on T^*M

- The group $C^\infty(M, \mathbf{R})$ acts on T^*M by $\underline{f}(q, p) = (q, p + df(q))$, $f \in C^\infty(M, \mathbf{R})$. It preserves the 2-form $\omega = dp \wedge dq$.
- The moment map is

$$\mu \colon (q, p) \mapsto d[f \mapsto f(q)].$$

Note: $[f \mapsto f(q)] \in C^\infty(C^\infty(M, \mathbf{R}), \mathbf{R})$ is not invariant, but its differential is an invariant 1-form on $C^\infty(M, \mathbf{R})$.

- Also $\mu(q, p) = d\delta_q$, where δ_q is the Dirac function. The moment map is the differential of a distribution.

Action of $C^\infty(\Sigma, \mathbf{R})$ on $\Omega^1(\Sigma)$

- The group $C^\infty(\Sigma, \mathbf{R})$ acts on $\Omega^1(\Sigma)$, preserving the 2-form $\omega(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta$.
- For all $f \in C^\infty(\Sigma, \mathbf{R})$, $\alpha \in \Omega^1(\Sigma)$, $\underline{f}(\alpha) = \alpha + df$.
- The moment map is:

$$\mu \colon \alpha \mapsto d\left[f \mapsto \int_\Sigma f\, d\alpha\right].$$

- Again, the moment map is the differential of a distribution: $[d\alpha] : f \mapsto \int_\Sigma f\, d\alpha$. Heuristically, people used to think of $d\alpha$ as the curvature, but here it assumes its true nature.

Action of $\mathrm{Diff}(\Sigma)$ on $\Omega^1(\Sigma)$

- Σ is a oriented surface.
- $\Omega^1(\Sigma)$ is the space of 1-forms on Σ, $\alpha, \beta \in \Omega^1(\Sigma)$.

$$\omega(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta.$$

- Group $\mathrm{Diff}^+(\Sigma)$, action:

$$\phi \in \mathrm{Diff}^+(\Sigma), \alpha \in \Omega^1(\Sigma), \quad \underline{\phi}(\alpha) = \phi_*(\alpha).$$

- The moment map:

$$\mu(\alpha)(P)_r(\delta r) = \frac{1}{2} \int_\Sigma \alpha \wedge P(r)^* \left(\frac{\partial P(r)_*(\alpha)}{\partial r}(\delta r) \right),$$

Action of $\Omega^1(\Sigma)$ on itself

- The group $\Omega^1(\Sigma)$ acts additively on itself, preserving the 2-form $\omega(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta$.
- The moment map is:

$$\mu \colon \alpha \mapsto d\left[\beta \mapsto \int_\Sigma \alpha \wedge \beta\right].$$

- Here again, the moment map is the differential of a distribution.
- The space $\Omega^1(\Sigma)$ is symplectic in the sense above.
    1. The group of automorphisms is transitive.
    2. The moment map μ is injective.

Immersing S^1 in $\mathbf{R}^2$

- We consider $\mathrm{Imm}(S^1, \mathbf{R}^2)$ and $\omega = d\alpha$, with

$$\alpha(\delta x) = \int_0^{2\pi} \frac{1}{\|\dot{x}(t)\|^2} \langle \ddot{x}(t) | \delta\dot{x}(t) \rangle \, dt, \;\; x \in \mathrm{Imm}(S^1, \mathbf{R}^2).$$

  $\mathrm{Diff}^+(S^1)$ acts on $\mathrm{Imm}(S^1, \mathbf{R}^2)$ by $\underline{\varphi}(x) = x \circ \varphi^{-1}$.

- On the connected component of the standard immersion $t \mapsto (\cos(t), \sin(t))$, the moment map is, up to a constant:

$$\mu(x)(P)_r(\delta r) = \int_0^{2\pi} \left\{ \frac{\|x''(u)\|^2}{\|x'(u)\|^2} - \frac{d^2}{du^2} \log \|x'(u)\|^2 \right\} \delta u \, du.$$

  $P \colon r \mapsto \varphi$ is a n-plot of $\mathrm{Diff}_+(S^1)$, $r \in \mathrm{dom}(P)$, $\delta r \in \mathbf{R}^n$, $u = \varphi^{-1}(t)$, where t is the parameter of $x \in \mathrm{Imm}(S^1, \mathbf{R}^2)$, and $\delta u = D(r \mapsto u)(r)(\delta r)$.

Immersing S^1 in $\mathbf{R}^2$

- The affine cocycle (lack of equivariance) of the $\mathrm{Diff}_+(S^1)$ action on $\mathrm{Imm}(S^1, \mathbf{R}^2)$ are cohomologous to θ defined by,

$$\theta(g)(P)_r(\delta r) = \int_0^{2\pi} \frac{3\gamma''(u)^2 - 2\gamma'''(u)\gamma'(u)}{\gamma'(u)^2}\, \delta u \, du,$$

  where $g \in \mathrm{Diff}^+(S^1)$ and $\gamma = g^{-1}$. The integrand of the right-hand side is the Schwarzian derivative.
- The cocycle θ of this integral construction of the moment map in diffeology, extends the Souriau's cocycle of symplectic geometry.

Symplectic Reduction

Reduction of a Contact Manifold

Let us recall that a contact form on a manifold is a differential form λ such that $\ker(\lambda) \cap \ker(d\lambda) = \{0\}$. The characteristics of $d\lambda$ are the integral curves of the Reeb vector field ξ defined uniquely by $\lambda(\xi) = 1$ and $d\lambda(\xi) = 0$.

Theorem. *Let λ be a contact form on a manifold Y. There always exists on the space $\mathcal{S}$ of charateristics of $d\lambda$ a parasymplectic form $\omega \in \Omega^2(\mathcal{S})$ such that*

$$\mathrm{class}^*(\omega) = d\lambda, \ \textit{with } \mathrm{class} : Y \to \mathcal{S} = Y/\ker(d\lambda).$$

When $\mathcal{S}$ is a manifold then ω is symplectic. Moreover, if Y has dimension $2n+1$ then $\mathcal{S}$ has dimension $2n$ and is symplectically generated.

Example of the Geodesics of T^2

The spaces of geodesic trajectories are parasymplectic as a particular case of the reduction of a contact manifold. For example, the geodesic trajectories of $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ are the projections of the affine lines in $\mathbf{R}^2$. The space $\mathrm{Geod}(T^2)$ is then the quotient of $S^1 \times \mathbf{R}$ by

$$(u', \rho') \sim (u, \rho) \Leftrightarrow u' = u \text{ and } \rho' = \rho + na + mb,$$

with $u = (a, b)$, the direction vector and $(n, m) \in \mathbf{Z}^2$. It is a quasifold $\mathbf{R}^2/\mathbf{Z}^3$, fibered on S^1 with fiber T_u over u, a rational or irrational torus depending on the rationality of u:

$$\pi : \mathrm{Geod}(T^2) \to S^1, \text{ with } \pi^{-1}(u) = T_u \text{ and } T_u \simeq \mathbf{R}/(a\mathbf{Z} + b\mathbf{Z}).$$

Singular Reduction in Infinite Dimension

Example of $\mathcal{E} = \{(f_n)_{n\in\mathbf{Z}} \mid f_n \downarrow 0\}$, representing smooth complex periodics functions, with $\omega = \frac{1}{\pi}\int_0^1 \hat{x}^*(\text{surf})\,dx$, and $\mathbf{R}$ acting on $\mathcal{E}$ by $\underline{t}\big((f_n)_{n\in\mathbf{Z}}\big) = \big(e^{2i\pi\alpha_n t} f_n\big)_{n\in\mathbf{Z}}$, with α_n independant on $\mathbf{Q}$.

- Moment maps: $h(f) = \mathrm{E}(f)dt$, $\mathrm{E}(f) = \sum_{n\in\mathbf{Z}} \alpha_n \|f_n\|^2 + c$.
- Let $\mathcal{S}^\infty_\alpha = \mathrm{E}^{-1}(1)$ $(c = 0)$. The singular orbits of $\mathbf{R}$ on $\mathcal{S}^\infty_\alpha$ are the harmonics S^1_k with $f_n = 0$ if $n \neq k$. They are circles of radius $1/\sqrt{\alpha_k}$. The other orbits are principal and diffeomorphic to $\mathbf{R}$.
- Call quasi-projective space the quotient $\mathbf{C}\mathrm{P}^\infty_\alpha = \mathcal{S}^\infty_\alpha/\mathbf{R}$. The form $\omega \restriction \mathcal{S}^\infty_\alpha$ passes to $\mathbf{C}\mathrm{P}^\infty_\alpha$ into a parasympectic form ϖ, despite the infinitely many singular orbits.

Toric Quasifolds as Parasymplectic Spaces

One can embed any space $\mathbf{C}^N$ into $\mathcal{E} = \{(f_n)_{n\in\mathbf{Z}} \mid f_n \downarrow 0\}$ by $(Z_1, \ldots, Z_N) \mapsto (0_\infty, Z_1, \ldots, Z_N, 0_\infty)$, and let $\mathbf{R}$ acting on $\mathbf{C}^N \subset \mathcal{E}$ by some sub-action of $\underline{t}\big((f_n)_{n\in\mathbf{Z}}\big) = \big(e^{2i\pi\alpha_n t} f_n\big)_{n\in\mathbf{Z}}$. By restriction of the infinity dimension case, one gets the constructions of Prato's quasispheres. In this sense $\mathcal{E}$ is the total classifying space for quasiprojective spaces.

This 1-dimensionsional real action can be extended to any irrational action of $\mathbf{R}^k$ on $\mathcal{E}$. This will make $\mathcal{E}$ as the total classifying space for general toric quasifolds, which are symplectically generated diffeological spaces, including toric manifolds and toric orbifolds.

Prequantization

Integration Bundles on Parasymplectic Manifolds

Let (M, ω) be a parasymplectic manifold. Define the group of periods and the torus of periods of ω by:

$$P_\omega = \left\{ \int_\sigma \omega \mid \sigma \in H_2(M, \mathbf{Z}) \right\}, \text{ and } T_\omega = \mathbf{R}/\,P_\omega.$$

Theorem. *There exists a* T_ω*-principal bundle* $\pi : Y \to M$*, equipped with a connection form* λ *of curvature* ω*. That is:*

$$\pi^*(\omega) = d\lambda.$$

Such integration bundles are classified, up to equivalence, by $\mathrm{Ext}(H_1(M, \mathbf{Z}), P_\omega)$.

When ω is not integral, $P_\omega \neq a\mathbf{Z}$, then T_ω is an irrational torus and Y is a diffeological space but not a manifold.

Generalized Prequantum Bundles on Parasymplectic Spaces

Let (X, ω) be a simply connected parasymplectic diffeological space. Let $x \in X$ and $\hat{x}$ be the constant loop. Let P_ω and T_ω be the group and the torus of periods of $K\omega \upharpoonright \mathrm{Loops}(X)$:

$$P_\omega = \left\{ \int_\sigma K\omega \mid \sigma \in \mathrm{Loops}(\mathrm{Loops}(X, x), \hat{x}) \right\} \text{ and } T_\omega = \mathbf{R}/P_\omega.$$

Define on $\mathrm{Paths}(X, x)$ the equivalence relation $\gamma \sim \gamma'$ if:

$$\gamma(1) = \gamma'(1) \text{ and } \int_{\hat{x}}^{\gamma \vee \bar{\gamma}'} K\omega \in P_\omega, \quad \text{with } \bar{\gamma}(t) = \gamma(1-t).$$

Theorem. *The quotient* $Y = \mathrm{Paths}(X, x)/\sim$ *is a* T_ω*-principal bundle, for the concatenation with loops and projection* $\pi : \mathrm{class}(\gamma) \mapsto \gamma(1)$*. The* 1*-form* $K\omega$ *projects onto* Y *in a connection form* λ *of curvature* ω*.*

Generalized Prequantum Bundles on Parasymplectic Spaces

In conclusion: For any simply connected parasymplectic diffeological spaces[3] (finite or infinite dimensional, with or without singularities), there is a (unique up to equivalence in this case) integration bundle which is a quotient of a space of paths. It can also be called a generalized prequantum bundle.

$$
\begin{array}{ccc}
\mathrm{Paths}(X, x) & \xrightarrow{\quad \text{class} \quad} & Y, \lambda \\
 & {\scriptstyle \hat{\mathbf{I}}} \searrow \quad \swarrow {\scriptstyle \pi} & \\
 & X, \omega &
\end{array}
$$

[3] The non simply connected case is a work in progress.

Global Analysis

Hamiltonian Diffeomorphisms

Theorem. *For any (connected) parasymplectic space* (X, ω)*, there exists a largest connected subgroup* $\mathrm{Ham}(X, \omega)$ *in* $G_\omega = \mathrm{Diff}(X, \omega)$ *whose holonomy is trivial. This is the group of Hamiltonian diffeomorphisms.*

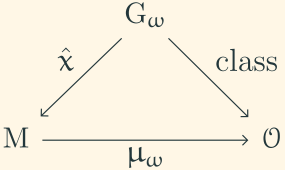
Proof. Let $\tilde{G}^\circ_\omega$ be the universal covering of the identity component of G_ω. Every element γ of the holonomy group Γ_ω is a closed 1-form on G_ω. Let $k(\gamma)$ be the real homomorphism on $\tilde{G}^\circ_\omega$ such that $\pi^*(\gamma) = d[k(\gamma)]$. Let

$$\hat{H}_\omega = \bigcap_{\gamma \in \Gamma_\omega} \ker\big(k(\gamma)\big), \text{ then } \mathrm{Ham}(X, \omega) = \pi(\hat{H}^\circ_\omega),$$

with $\pi : \tilde{G}^\circ_\omega \to G^\circ_\omega$. □

Symplectic Manifolds are Coadjoint Orbits

The group $G_\omega = \mathrm{Diff}(M, \omega)$ of a symplectic manifold is transitive. The orbit map $\hat{x} : \varphi \mapsto \varphi(x)$, $x \in M$, is a principal fiber bundle with group G_ω^x, in particular a subduction.



The universal moment map μ_ω is injective (ω symplectic). Let $\mathcal{O} = G_\omega / \mathrm{Stab}(\epsilon)$, with $\epsilon = \mu_\omega(x)$ and $\mathrm{class} : G_\omega \to \mathcal{O}$.

Then, $\mu_\omega : M \to \mathcal{O}$ is an equivariant diffeomorphism for the, possibly affine, coadjoint action $\mathrm{Ad}_* + \theta_\omega$.

Example: the torus T^2 is an affine coadjoint orbit of itself.

Symplectic Manifolds are (Linear) Coadjoint Orbits

An integration bundle (Y, λ) of (M, ω) produces a central extension of the Hamiltonian diffeomorphisms:

$$\mathbf{1} \longrightarrow \mathrm{T}_\omega \longrightarrow \mathrm{Aut}(\mathrm{Y}, \lambda)^\circ \xrightarrow{\mathrm{pr}} \mathrm{Ham}(\mathrm{X}, \omega) \longrightarrow \mathbf{1}$$

$\mathcal{A}^*$ and $\mathcal{H}^*_\omega$ denote the respective spaces of momenta. From here one gets the commutative diagram of moment maps:

$$\begin{array}{ccc} \mathrm{Y} & \xrightarrow{\mu_\mathrm{Y}} & \mathcal{A}^* \\ {\scriptstyle \pi}\downarrow & \overset{\bar\mu_\mathrm{M}}{\nearrow} & \uparrow{\scriptstyle \mathrm{pr}^*} \\ \mathrm{M} & \xrightarrow[\mu_\mathrm{M}]{} & \mathcal{H}^*_\omega \end{array}$$

The moment map $\bar\mu_\mathrm{M}$, which is the projection of the moment map μ_Y, identifies M with a linear-coadjoint (non affine) orbit of $\mathrm{Aut}(\mathrm{Y}, \lambda)$.

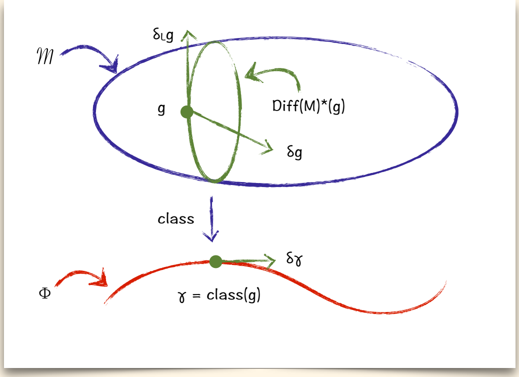
General Relativity and General Covariance

Diffeology can be used to formalize and give a rigorous formulation to Souriau's General Covariance Principle :

- Passive matter are pointed 1-forms on the quotient Φ of the pseudo-Riemannian metrics $\mathcal{M}$ on space-time M, by the group of compactly supported diffeomorphisms $\mathrm{Diff}_c(M)$.
- Active matter, i.e. Einstein's fields equations, appears as a closed 1-form on Φ.

This heuristic model "Modèle de particule à spin dans le champ électromagnétique et gravitationnel" (1974), had yet never been founded within a rigorous framework.

General Relativity and General Covariance



The Physis $\Phi = \mathcal{M}/\mathrm{Diff}_c(M)$.

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